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Further results on mean graphs

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Abstract Let $G(V, E)$ be a graph with p vertices and q edges. For every assignment $f : V(G) \rightarrow \{0, 1, 2, 3, \ldots, q\}$, an induced edge labeling $f^* : E(G) \rightarrow \{1, 2, 3 \ldots, q\}$ is defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2}, & \text{if } f(u) \text{ and } f(v) \text{ are of the same parity,} \\ \frac{f(u)+f(v)+1}{2}, & \text{otherwise.} \end{cases}$$

for every edge $uv \in E(G)$. If $f^*(E) = \{1, 2, \ldots, q\}$, then we say that f is a mean labeling of G. If a graph G admits a mean labeling, then G is called a mean graph. In this paper we study the meanness of the splitting graph of the path P_n and $C_{2n}(n \geq 2)$, meanness of some duplicate graphs, meanness of Armed crown, meanness of Bi-armed crown and mean labeling of cyclic snakes.

Keywords Mean labeling, splitting graphs, duplicate graphs, armed crown, cyclic snake.

§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology we follow [1].

Path on n vertices is denoted by P_n and cycle on n vertices is denoted by C_n. $K_{1,m}$ is called a star and it is denoted by S_m. The graph $K_2 \times K_2 \times K_2$ is called the cube and it is denoted by Q_3. The Union of two graphs G_1 and G_2 is the graph $G_1 \mathrm{U} G_2$ with $V(G_1 \mathrm{U} G_2) = V(G_1) \mathrm{U} V(G_2)$ and $E(G_1 \mathrm{U} G_2) = E(G_1) \mathrm{U} E(G_2)$. The union of m disjoint copies of a graph G is denoted by mG. The $H-$graph of a path P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ by an edge if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}}$ if n is even.

A vertex labeling of G is an assignment $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$. For a vertex labeling f, the induced edge labeling $f^\star$ is defined by

$f^*(uv) = \frac{f(u)+f(v)}{2}$ for any edge uv in G, that is,

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2}, & \text{if } f(u) \text{ and } f(v) \text{ are of same parity,} \\ \frac{f(u)+f(v)+1}{2}, & \text{otherwise.} \end{cases}$$

A vertex labeling f is called a mean labeling of G if its induced edge labeling $f^* : E \to \{1, 2, \ldots, q\}$ is a bijection, that is, $f^*(E) = \{1, 2, \ldots, q\}$. If a graph G has a mean labeling, then we say that G is a mean graph.

The mean labeling of the following graph is given in Figure 1.

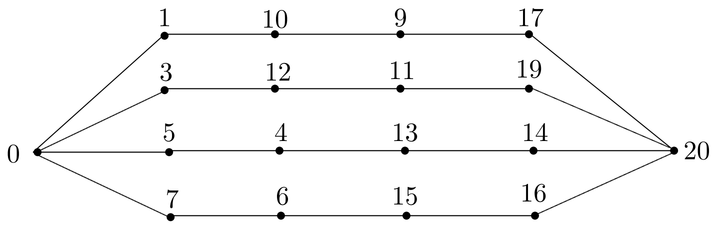


Figure 1.

The concept of mean labeling was introduced by S. Somasundaram and R. Ponraj in [4]. In [2, 4, 5], they have studied the mean labeling of some standard graphs. Also some standard results are proved in [3, 6].

In this paper, we have established the meanness of the splitting graph of the path and the cycle C_{2n} for $n \geq 2$. Also we discuss about the meanness of some duplicate graphs and the meanness of Armed crowns, Bi-armed crowns and cyclic snakes.

We use the following results in the subsequent theorems:

Theorem 1.[4] The cycle C_n is a mean graph, $n \geq 3$.

Theorem 2.[5] If $p > q + 1$, then the (p, q) graph G is not a mean graph.

§2. Meanness of the splitting graph

Let G be a graph. For each point v of a graph G, take a new point v'. Join v' to those points of G adjacent to v. The graph thus obtained is called the splitting graph of G. We denote it by $S'(G)$.

Here we prove the meanness of the splitting graph of the path P_n for $n \geq 2$ and the cycle C_{2n} for $n \geq 2$.

Theorem 3. $S'(P_n)$ is a mean graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n and $v_1, v_2, \ldots, v_n,\ v'_1, v'_2, \ldots, v'_n$ be the vertices of $S'(P_n)$. $S'(P_n)$ has $2n$ vertices and $3(n-1)$ edges.

We define $f : V((S'(P_n)) \to \{0, 1, 2, \ldots, q\}$ by

$$f(v_{2i+1}) = 2i, 0 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor,$$

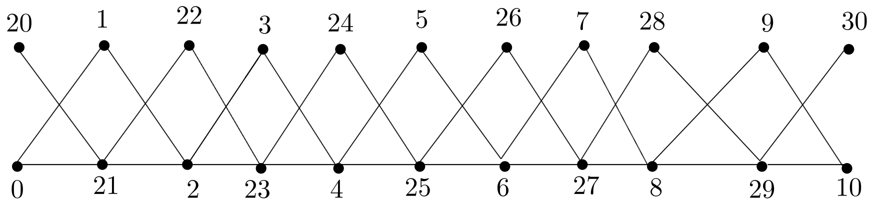
$$f(v_{2i}) = 2n - 1 + 2(i-1), 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor,$$

$$f(v'_{2i+1}) = 2n - 2 + 2i, 0 \le i \le \left\lfloor \frac{n-1}{2} \right\rfloor,$$

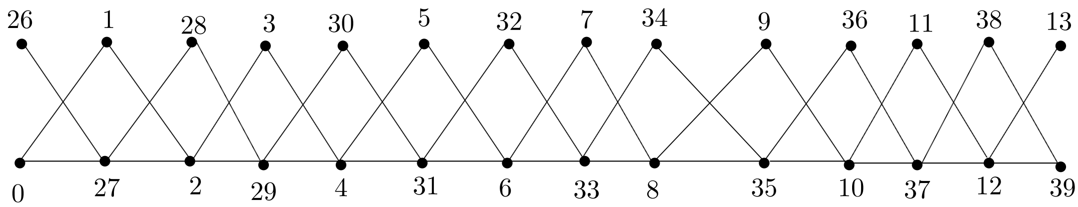
$$f(v'_{2i}) = 2i - 1, 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor.$$

It can be verified that the label of the edges of $S'(P_n)$ are $1, 2, \ldots, q$ and hence $S'(P_n)$ is a mean graph.

For example, the mean labelings of $S'(P_{11})$ and $S'(P_{14})$ are shown in Figure 2.



$S'(P_{11})$



$S'(P_{14})$

Figure 2.

Theorem 4. $S'(C_{2n})$ is a mean graph.

Proof. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of the cycle C_{2n} and $v_1, v_2, \ldots, v_{2n}, v'_1, v'_2, \ldots, v'_{2n}$ be the vertices of $S'(C_{2n})$.

Note that $S'(C_{2n})$ has $4n$ vertices and $6n$ edges.

Now we define $f : V(S'(C_{2n})) \to \{0, 1, 2, \ldots, q\}$ as follows:

$$f(v_{2i+1}) = \begin{cases} 4i, & 0 \le i \le \lfloor \frac{n}{2} \rfloor, \\ 5 + 4((n-1) - i), & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n - 1. \end{cases}$$

$$f(v_{2i}) = \begin{cases} 4n + 2 + 4i - 4, & 1 \le i \le \lceil \frac{n}{2} \rceil, \\ 4n + 3 + 4(n - i), & \lceil \frac{n}{2} \rceil + 1 \le i \le n. \end{cases}$$

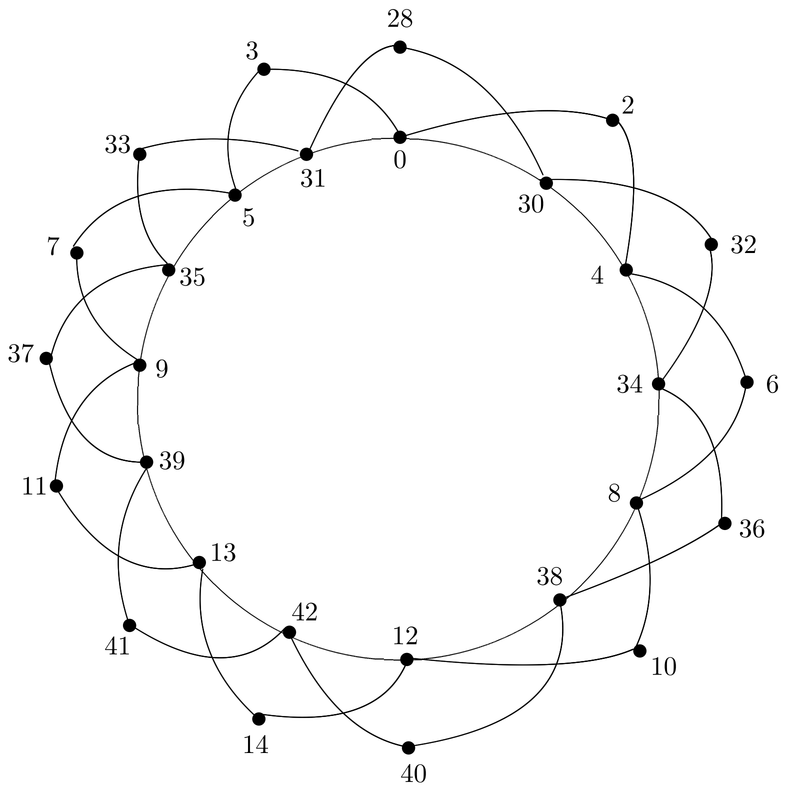
$$f(v'_{2i+1}) = \begin{cases} 4n + 4i, & 0 \le i \le \lfloor \frac{n}{2} \rfloor, \\ 4n + 5 + 4((n-1) - i), & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n - 1. \end{cases}$$

$$f(v'_{2i}) = \begin{cases} 2+4i-4, & 1 \le i \le \lceil \frac{n}{2} \rceil, \\ 3+4(n-i), & \lceil \frac{n}{2} \rceil + 1 \le i \le n. \end{cases}$$

It can be verified that the labels of the edges of $S'(C_{2n})$ are $1, 2, 3, \ldots, q$.

Hence $S'(C_{2n})$ is a mean graph.

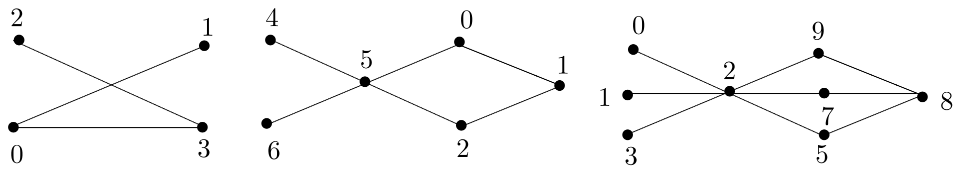
For example, the mean labeling of $S'(C_{14})$ is shown in Figure 3.



$S'(C_{14})$

Figure 3.

The mean labelings of the splitting graph of $K_{1,1}$, $K_{1,2}$ and $K_{1,3}$ are shown in the following Figure 4.



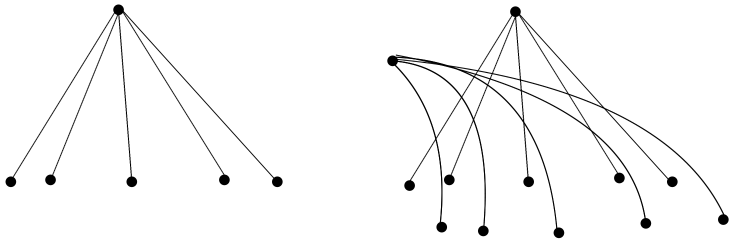
$S'(K_{1,1})$ $S'(K_{1,2})$ $S'(K_{1,3})$

Figure 4.

§3. Meanness of Duplicate Graphs

Let G be a graph with $V(G)$ as vertex set. Let V' be the set of vertices with $|V'| = |V|$. For each point $a \in V$, we can associate a unique point $a' \in V'$. The duplicate graph of G denoted by $D(G)$ has vertex set $V \cup V'$. If a and b are adjacent in G then $a'b$ and ab' are adjacent in $D(G)$.

For example, $D(S_5) = 2S_5$ is shown in the following Figure 5.



S_5 $\qquad$ $D(S_5) = 2S_5$

Figure 5.

In this section we characterize the meanness of some Duplicate graphs.

Theorem 5. Duplicate graph of a path is not a mean graph.

Proof. Let P_n be a path. $D(P_n) = 2P_n$. $D(P_n)$ is disconnected and it has $2n$ vertices and $2n-2$ edges. Therefore $D(P_n)$ is not a mean graph for any n by Theorem 2.

Theorem 6. The disconnected graph $2C_n$ for $n \geq 3$ is a mean graph.

Proof. The graph $2C_n$ has $2n$ vertices and $2n$ edges. Let $v_1^1, v_2^1, \ldots, v_n^1$ be the vertices of the first copy of C_n and $v_1^2, v_2^2, \ldots, v_n^2$ be the vertices of the second copy of C_n.

We define $f : V(2C_n) \to \{0, 1, 2, \ldots, 2n\}$ as follows:

Case (i). Suppose n is odd say $n = 2k+1$.

$$f(v_i^1) = \begin{cases} i-1, & 1 \leq i \leq k, \\ i, & k+1 \leq i \leq 2k. \end{cases}$$

$f(v_n^1) = n+1$,

$$f(v_i^2) = \begin{cases} n+2(i-1), & 1 \leq i \leq k+1, \\ 2n-2(i-(k+2)), & k+2 \leq i \leq 2k+1. \end{cases}$$

Case (ii). Suppose n is even say $n = 2k$.

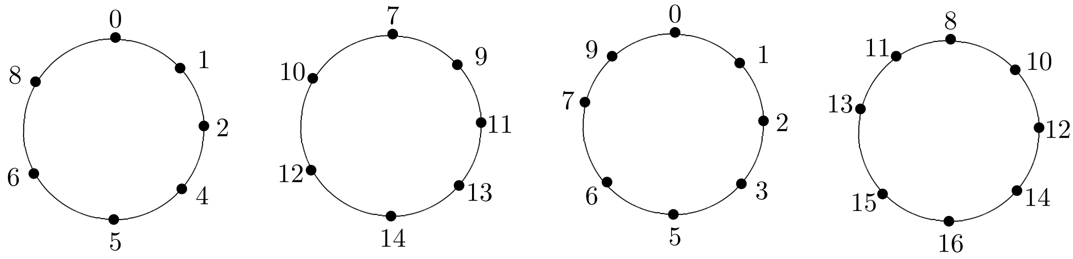
$$f(v_i^1) = \begin{cases} i-1, & 1 \leq i \leq k, \\ i, & k+1 \leq i \leq 2k-1. \end{cases}$$

$f(v_n^1) = n+1$

$$f(v_i^2) = \begin{cases} n+2(i-1), & 1 \leq i \leq k+1, \\ 2n-1-2(i-(k+2)), & k+2 \leq i \leq 2k. \end{cases}$$

It can be verified that the label of the edges of the graph are $1, 2, 3, \ldots, 2n$.

Hence $2C_n$ for $n \geq 3$ is a mean graph. For example the mean labelings of $2C_7$ and $2C_8$ are shown in Figure 6.



$2C_7$ $2C_8$

Figure 6.

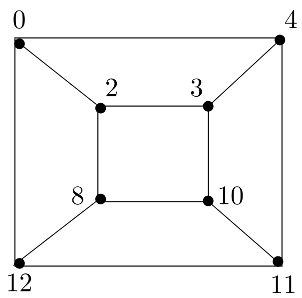
Corollary 1. Duplicate graph of the cycle C_n is a mean graph.

Proof. $D(C_n) = C_{2n}$ when n is odd. But C_{2n} is a mean graph by Theorem 1 and $D(C_n) = 2C_n$ when n is even. $2C_n$ is a mean graph by Theorem 6. Therefore $D(C_n)$ is a mean graph.

Theorem 7. mQ_3 is a mean graph.

Proof. For $1 \leq j \leq m$, let $v_1^j, v_2^j, \ldots, v_8^j$ be the vertices in the j^{th} copy of Q_3. The graph mQ_3 has $8m$ vertices and $12m$ edges. We define $f : V(mQ_3) \to \{0, 1, 2, \ldots, 12m\}$ as follows.

When $m = 1$, label the vertices of Q_3 as follows:



For $m > 1$, label the vertices of mQ_3 as follows:

$$\begin{aligned}
f(v_1^j) &= 12(j-1), \qquad 1 \leq j \leq m \\
f(v_i^j) &= 12(j-1) + i, \quad 2 \leq i \leq 4, 1 \leq j \leq m \\
f(v_5^j) &= 12(j-1) + 8, \quad 1 \leq j \leq m \\
f(v_6^j) &= 12(j-1) + 9, \quad 1 \leq j \leq m-1 \\
f(v_7^j) &= 12(j-1) + 11, \quad 1 \leq j \leq m-1 \\
f(v_8^j) &= 12j + 1, \qquad 1 \leq j \leq m-1
\end{aligned}$$

$f(v_6^m) = 12m - 2$, $f(v_7^m) = 12m - 1$ and $f(v_8^m) = 12m$. It can be easily verified that the label of the edges of the graph are $1, 2, 3, \ldots, 12m$.

Then mQ_3 is a mean graph. For example, the mean labeling of $3Q_3$ is shown in Figure 7.

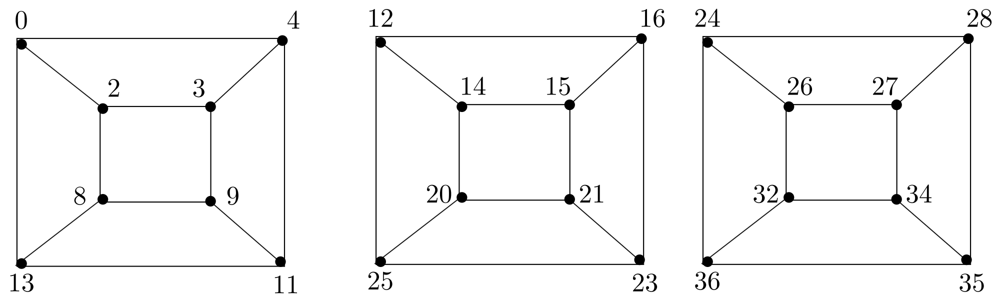


Figure 7.

Corollary 2. Duplicate Graph of Q_3 is a mean graph.

Proof. $D(Q_3) = 2Q_3$ which is a mean graph by Theorem 7.

Theorem 8. Let Q be the quadrilateral with one chord. Duplicate graph of Q is a mean graph.

Proof. The following is a mean labeling of $D(Q)$.

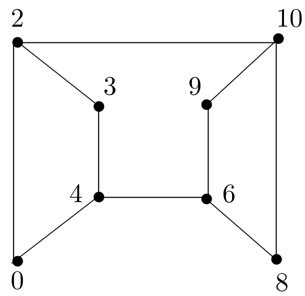


Figure 8.

Theorem 9. Duplicate graph of a $H-$graph is not a mean graph.

Proof. Let G be a $H-$graph on $2n$ vertices. $D(G) = 2G$. $D(G)$ is disconnected and it has $4n$ vertices and $4n - 2$ edges. Therefore $D(G)$ is not a mean graph by Theorem 2.

By Theorem 2 we have the following result.

Theorem 10. For any tree $T, D(T) = 2T$ which is not a mean graph.

§4. Meanness of special classes of graphs

Armed crowns are cycles attached with paths of equal lengths at each vertex of the cycle. We denote an armed crown by $C_n \ominus P_m$ where P_m is a path of length $m - 1$.

Theorem 11. $C_n \ominus P_m$ is a mean graph for $n \geq 3$ and $m \geq 2$.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle C_n. Let $v_j^1, v_j^2, \ldots v_j^m$ be the vertices of P_m attached with u_i by identifying v_j^m with u_j for $1 \leq j \leq\ n$.

The graph $C_n \ominus P_m$ has mn edges and mn vertices.

Case (i) $n \equiv 0 (\text{mod } 4)$

Let $n = 4k$ for some k. we define $f : V(C_n \ominus P_m) \to \{0, 1, 2, \ldots, q = mn\}$ as follows.

For $1 \leq i \leq m$,

$$f(v_j^i) = \begin{cases} i-1+(j-1)m, & \text{if } j \text{ is odd, } 1 \leq j \leq 2k, \\ m-i+(j-1)m, & \text{if } j \text{ is even, } 1 \leq j \leq 2k, \\ i+(j-1)m, & \text{if } j \text{ is odd, } 2k+1 \leq j \leq 4k-1, \\ m+1-i+(j-1)m, & \text{if } j \text{ is even, } 2k+1 \leq j \leq 4k-1. \end{cases}$$

$f(v_{4k}^i) = (4k-1)m + 2i - 1, \quad 1 \leq i \leq \lceil \frac{m}{2} \rceil.$
$f(v_{4k}^{m+1-i}) = (4k-1)m + 2i, \quad 1 \leq i \leq \lfloor \frac{m}{2} \rfloor.$

It can be verified that the label of the edges of $C_n \ominus P_m$ are $1, 2, 3, \ldots, mn$. Then f is a mean labeling of $C_n \ominus P_m$.

Case(ii) $n \equiv 1 (\text{mod } 4)$

Let $n = 4k+1$ for some k. we define $f : V(C_n \ominus P_m) \to \{0, 1, 2, \ldots, q = mn\}$ as follows.

For $1 \leq i \leq m$,

$$f(v_j^i) = \begin{cases} m-i+(j-1)m, & \text{if } j \text{ is odd, } 1 \leq j \leq 2k, \\ i-1+(j-1)m, & \text{if } j \text{ is even, } 1 \leq j \leq 2k, \\ m+1-i+(j-1)m, & \text{if } j \text{ is odd, } 2k+1 \leq j \leq 4k+1, \\ i+(j-1)m, & \text{if } j \text{ is even, } 2k+1 \leq j \leq 4k+1. \end{cases}$$

It is easy to check that the edge labels of $C_n \ominus P_m$ are $1, 2, 3, \ldots, q$ and hence $C_n \ominus P_m$ is a mean graph.

Case(iii) $n \equiv 2 (\text{mod } 4)$

Let $n = 4k+2$ for some k. we define $f : V(C_n \ominus P_m) \to \{0, 1, 2, \ldots, q = mn\}$ as follows.

For $1 \leq i \leq m$,

$$f(v_j^i) = \begin{cases} i-1+(j-1)m, & \text{if } j \text{ is odd, } 1 \leq j \leq 2k+1, \\ m-i+(j-1)m, & \text{if } j \text{ is even, } 1 \leq j \leq 2k+1, \\ i+(j-1)m, & \text{if } j \text{ is odd, } 2k+2 \leq j \leq 4k+1, \\ m+1-i+(j-1)m, & \text{if } j \text{ is even, } 2k+2 \leq j \leq 4k+1. \end{cases}$$

$f(v_{4k+2}^i) = (4k+1)m + 2i - 1, \quad 1 \leq i \leq \lceil \frac{m}{2} \rceil.$
$f(v_{4k+2}^{m+1-i}) = (4k+1)m + 2i, \quad 1 \leq i \leq \lfloor \frac{m}{2} \rfloor.$

It can be checked that the label of the edges of the given graph are $1, 2, 3, \ldots, mn$. Hence f is a mean labeling.

Case(iv) $n \equiv 3 (\text{mod } 4)$.

Let $n = 4k-1$, $k = 1, 2, 3 \ldots$, we define $f : V(C_n \ominus P_m) \to \{0, 1, 2, \ldots, q = mn\}$ as follows.

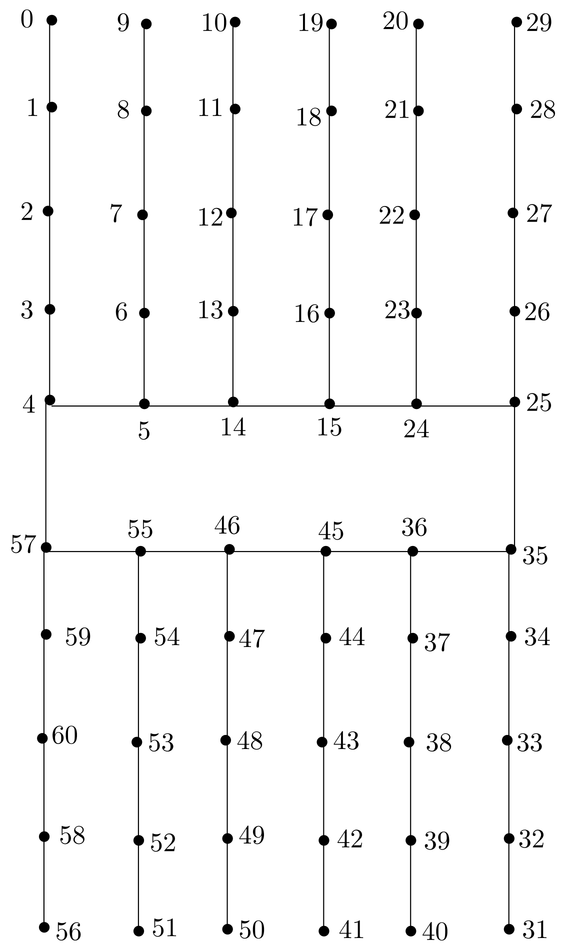
For $1 \leq i \leq m$,

$$f(v_j^i) = \begin{cases} m-i+(j-1)m, & \text{if } j \text{ is odd, } 1 \leq j \leq 2k-1, \\ i-1+(j-1)m, & \text{if } j \text{ is even, } 1 \leq j \leq 2k-1, \\ m+1-i+(j-1)m, & \text{if } j \text{ is odd, } 2k \leq j \leq 4k-1, \\ i+(j-1)m, & \text{if } j \text{ is even, } 2k \leq j \leq 4k-1. \end{cases}$$

It can be verified that the labels of the edges of $C_n \ominus P_m$ are $1, 2, 3, \ldots, q = mn$. Then f is clearly a mean labeling.

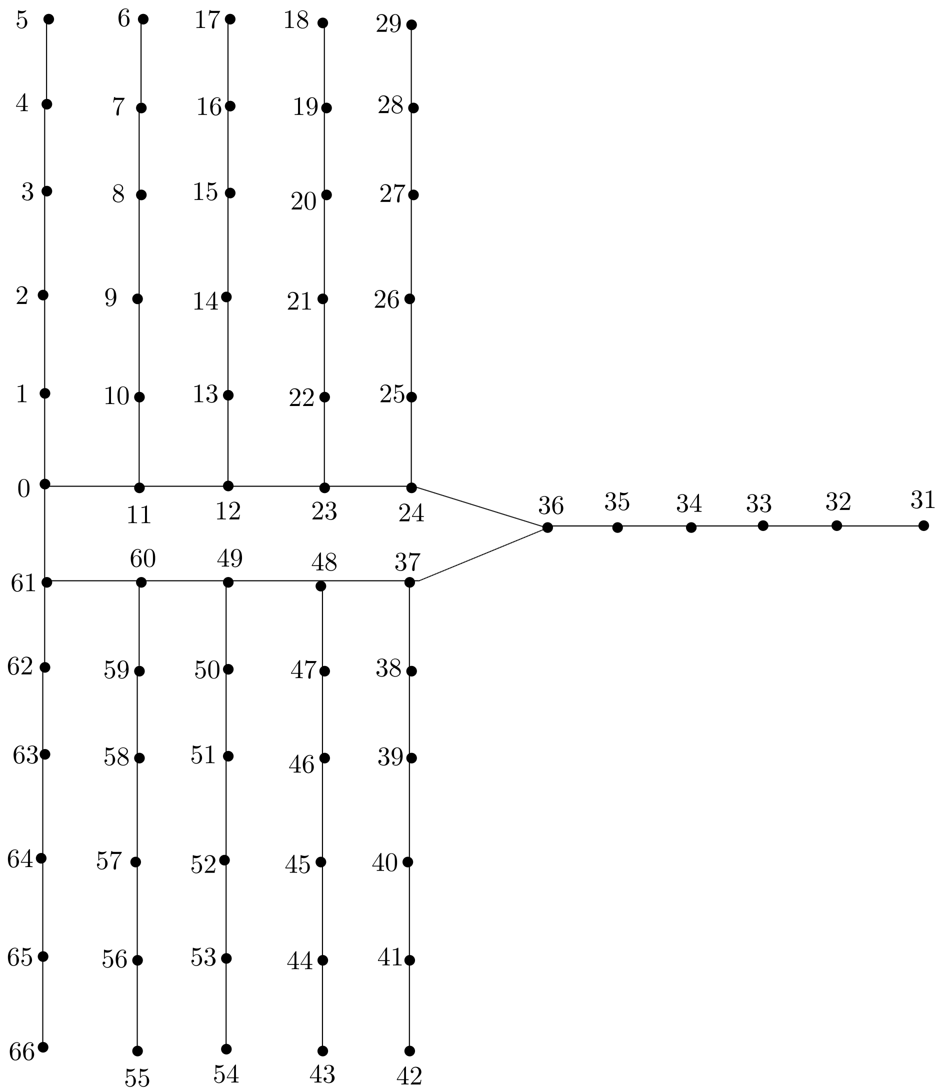
Hence $C_n \ominus P_m$ is a mean graph for $n \geq 3$ and $m \geq 1$.

For example the mean labelings of $C_{12} \ominus P_5$ and $C_{11} \ominus P_6$ are shown in Figure 9(a) and 9(b).



$C_{12} \ominus P_5$

Figure 9(a).



$C_{11} \ominus P_6$

Figure 9(b).

In the case of $m = 1$, $C_n \ominus P_m = C_n$ which is a mean graph by Theorem 1.

Bi-armed crown $C_n \ominus 2P_m$ is a graph obtained from a cycle C_n by identifying the pendent vertices of two vertex disjoint paths of same length $m - 1$ at each vertex of the cycle.

Theorem 12. The bi-armed crown $C_n \ominus 2P_m$ is a mean graph for all $n \geq 3$ and $m \geq 2$.

Proof. Let C_n be a cycle with vertices $u_1, u_2, \ldots, u_n$. Let $v_{j1}^1, v_{j1}^2, v_{j1}^3, \ldots, v_{j1}^m$ and $v_{j2}^1, v_{j2}^2, v_{j2}^3, \ldots, v_{j2}^m$ be the vertices of two vertex disjoint paths of length $m - 1$ in which the vertices $v_{j_1}^m$ and $v_{j_2}^m$ are identified with u_j for $1 \leq j \leq n$.

Case(i) n is odd. Let $n = 2k+1$ for some k. We define $f : V(C_n \ominus 2P_m) \to \{0, 1, 2, \ldots, q\}$

by

$$\begin{aligned}
f(v_{j1}^{i}) &= (i-1)+(2m-1)(j-1), \quad 1\le j\le k, 1\le i\le m.\\
f(v_{(k+1)1}^{i}) &= (i-1)+(2m-1)k, \quad 1\le i\le m-1.\\
f(v_{(k+1)1}^{m}) &= m+(2m-1)k.\\
f(v_{j1}^{i}) &= i+(2m-1)(j-1), \quad k+2\le j\le 2k+1, 1\le i\le m.
\end{aligned}$$

$$f(v_{j2}^{m+1-i}) = \begin{cases} i+m-2+(2m-1)(j-1), & 1\le j\le k,\ 1\le i\le m,\\ i+m-1+(2m-1)(j-1), & k+1\le j\le 2k+1,\ 1\le i\le m.\end{cases}$$

It can be verified that the label of the edges of $C_n \ominus 2P_m$ are $1, 2, 3\ldots, q$. Then f is a mean labeling.

Case(ii) n is even. Let $n = 2k$ for some k. We define $f : V(C_n \ominus 2P_m) \to \{0, 1, 2, \ldots, q\}$ as follows.

$$f(v_{j1}^{i}) = \begin{cases} i-1+(2m-1)(j-1), & 1\le j\le k,\ 1\le i\le m,\\ i+(2m-1)(j-1), & k+1\le j\le 2k-1,\ 1\le i\le m.\end{cases}$$

$f(v_{(2k)1}^{i}) = i+(2m-1)(2k-1), 1\le i\le m-1.$

$f(v_{(2k)1}^{m}) = i+1+(2m-1)(2k-1).$

$$f(v_{j2}^{m+1-i}) = \begin{cases} i+m-2+(2m-1)(j-1), & 1\le j\le k,\ 1\le i\le m,\\ i+m-1+(2m-1)(j-1), & k+1\le j\le 2k-1,\ 1\le i\le m.\end{cases}$$

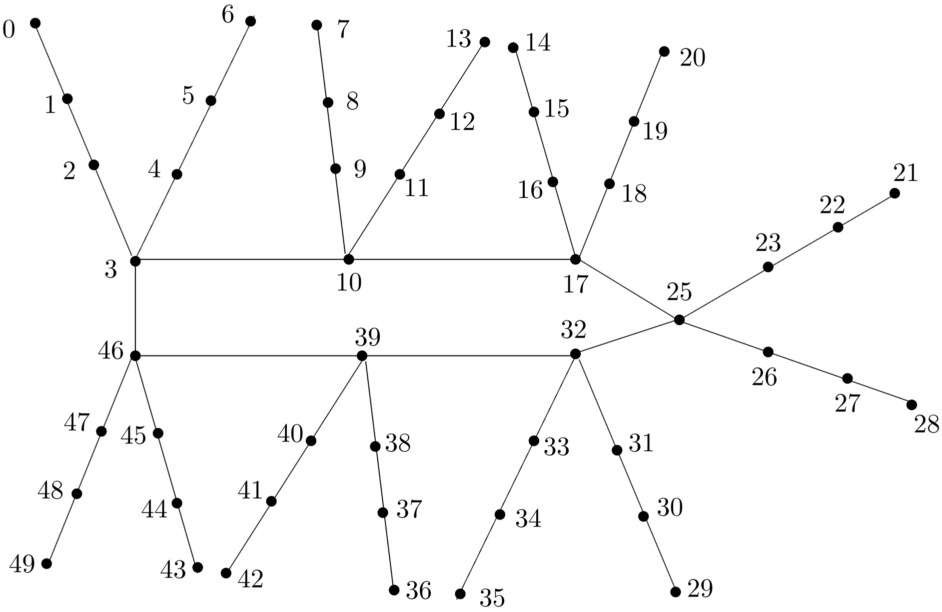
$f(v_{(2k)2}^{i}) = 2(i-1)+m+(2m-1)(2k-1), 1\le i\le \lceil m/2\rceil.$

$f(v_{(2k)2}^{m+1-i}) = 2i-1+m+(2m-1)(2k-1), 1\le i\le \lfloor m/2\rfloor.$

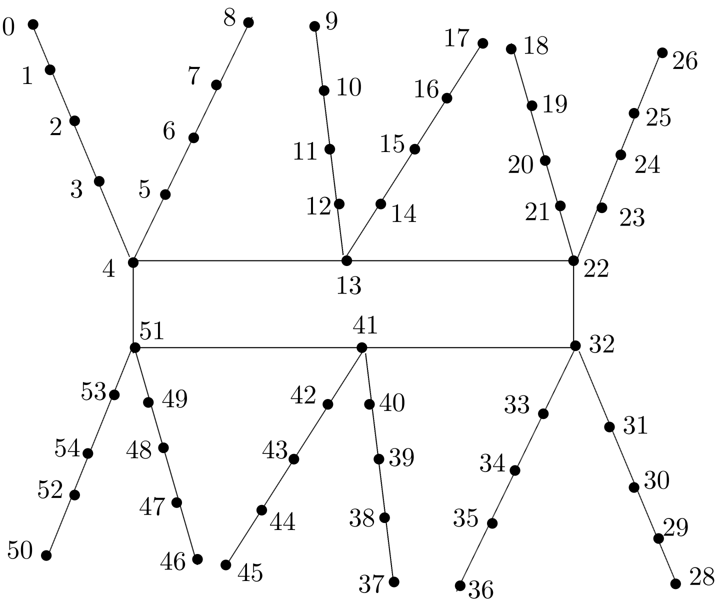
It is easy to check that the edge labels of $C_n \ominus 2P_m$ are $1, 2, 3, \ldots, q$. Then f is clearly a mean labeling.

Hence $C_n \ominus 2P_m$ is a mean graph for $n \ge 3$ and $m \ge 2$.

For example the mean labelings of $C_7 \ominus 2P_4$ and $C_6 \ominus 2P_5$ are shown in Figure 10.



$C_7 \ominus 2P_4$



$C_6 \ominus 2P_5$

Figure 10.

The cyclic snake mC_n is the graph obtained from m copies of C_n by identifying the vertex v_{k+2_j} in the j^{th} copy at a vertex $v_{1_{j+1}}$ in the $(j+1)^{th}$ copy when $n = 2k+1$ and identifying

the vertex v_{k+1_j} in the j^{th} copy at a vertex v_{1j+1} in the $(j+1)^{th}$ copy when $n = 2k$.

Theorem 13. The graph mC_n-snake, $m \geq 1$ and $n \geq 3$ has a mean labeling.

Proof. Let $v_{1_j}, v_{2_j}, \ldots, v_{n_j}$ be the vertices of mC_n for $1 \leq j \leq m$.

We prove this result by induction on m. Let $m = 1$. Label the vertices of C_n as follows:

Take

$$n = \begin{cases} 2k, & \text{if } n \text{ is even,} \\ 2k+1, & \text{if } n \text{ is odd.} \end{cases}$$

Then $f(v_{i_1}) = 2i - 2, 1 \leq i \leq k+1$

$$f(v_{(k+r)_1}) = \begin{cases} n - 2r + 3, \quad 2 \leq r \leq k, & \text{if } n \text{ is even,} \\ n - 2r + 4, \quad 2 \leq r \leq k+1, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore C_n is a mean graph.

Let $m = 2$. The cyclic snake $2C_n$ is the graph obtained from 2 copies of C_n by identifying the vertex $v_{(k+2)_1}$ in the first copy of C_n at a vertex v_{1_2} in the second of copy of C_n when $n = 2k+1$ and identifying the vertex $v_{(k+1)_1}$ in the first copy of C_n at a vertex v_{1_2} in the second copy of C_n when $n = 2k$.

Define a mean labeling g of $2C_n$ as follows:

$g(v_{i_1}) = f(v_{i_1}), 1 \leq i \leq n.$

$g(v_{i_2}) = f(v_{i_1}) + n, 2 \leq i \leq n.$

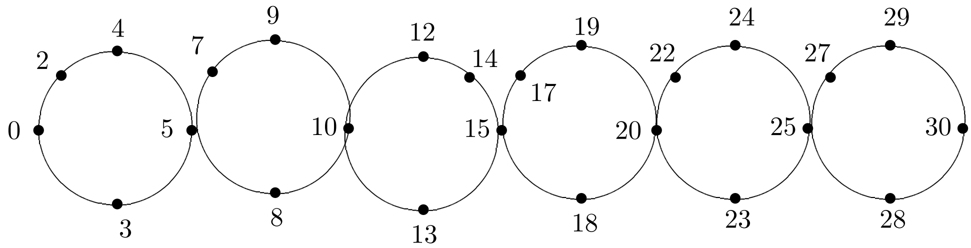
Thus $2C_n-$snake is a mean graph.

Assume that mC_n-snake is a mean graph for any $m \geq 1$. We will prove that $(m+1)C_n-$snake is a mean graph by giving a mean labeling of $(m+1)C_n$.

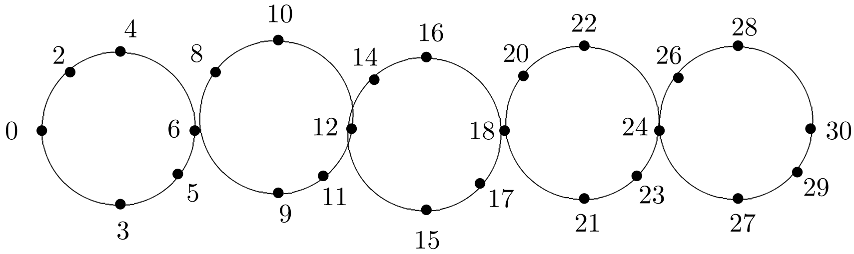
$$\begin{aligned} g(v_{i_1}) &= f(v_{i_1}), 1 \leq i \leq n, \\ g(v_{i_j}) &= f(v_{i_1}) + (j-1)n, \quad 2 \leq i \leq n, \quad 2 \leq j \leq m, \\ g(v_{i_m+1}) &= f(v_{i_1}) + mn, \quad 2 \leq i \leq n. \end{aligned}$$

Then the resultant labeling is a mean labeling of $(m+1)C_n-$snake. Hence the theorem.

For example, the mean labelings of $6C_5-$snake and $5C_6-$snake are shown in Figure 11.



$6C_5$-Snake



$5C_6$-Snake

Figure 11.

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A short interval result for the exponential divisor function

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Abstract An integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called the exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i} > 1$ if $b_i | a_i$ for every $i \in \{1, 2, \cdots, s\}$. Let $\tau^{(e)}(n)$ denote the number of exponential divisors of n, where $\tau^{(e)}(1) = 1$ by convention. The aim of this paper is to establish a short interval result for $-r$-th power of the function $\tau^{(e)}$ for any fixed integer $r \geq 1$.

Keywords The exponential divisor function, arithmetic function, short interval.

§1. Introduction

Let $n > 1$ be an integer of canonical from $n = \prod_{i=1}^{s} p_i^{a_i}$. An integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called the exponential divisor of n if $b_i | a_i$ for every $i \in \{1, 2, \cdots, s\}$, notation: $d|_e n$. By convention $1|_e 1$.

Let $\tau^{(e)}(n)$ denote the number of exponential divisors of n. The function $\tau^{(e)}$ is called the exponential divisor function. The properties of the function $\tau^{(e)}$ is investigated by several authors (see for example [1], [2], [3]).

Let $r \geq 1$ be a fixed integer and define $Q_r(x) := \sum_{n \leq x} (\tau^{(e)}(n))^{-r}$. Recently Chenghua Zheng [5] proved that the asymptotic formula

$$Q_r(x) = A_r x + x^{\frac{1}{2}} \log^{2^{-r}-2} (\sum_{j=0}^{N} d_j(r) \log^{-j} x + O(\log^{-N-1} x)) \tag{1}$$

holds for any fixed integer $N \geq 1$, where $d_0(r), d_1(r), \cdots, d_N(r)$ are computable constants, and

$$A_r := \prod_p (1 + \sum_{a=2}^{\infty} \frac{(\tau(a))^{-r} - (\tau(a-1))^{-r}}{p^a}). \tag{2}$$

The aim of this short note is to study the short interval case and prove the following.

Theorem. If $x^{\frac{1}{5}+2\epsilon} \leq y \leq x$, then

$$\sum_{x < n \leq x+y} (\tau^{(e)}(n))^{-r} = A_r y + (y x^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3\epsilon}{2}}), \tag{3}$$

where A_r is given by (2).

Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant. Suppose that $1 \le a \le b$ are fixed integers, the divisor function $d(a, b; k)$ is defined by

$$d(a, b; k) = \sum_{k=n_1^a n_2^b} 1.$$

The estimate $d(a, b; k) \ll k^{\epsilon^2}$ will be used freely. For any fixed $z \in \mathbb{C}$, $\zeta^z(s)(\Re s > 1)$ is defined by $\exp(z \log \zeta(s))$ such that $\log 1 = 0$.

§2. Proof of the theorem

Lemma 1. Suppose s is a complex number ($\Re s > 1$), then

$$F(s) := \sum_{n=1}^{\infty} \frac{(\tau^{(e)}(n))^{-r}}{n^s} = \zeta(s)\zeta^{2^{-r}-1}(2s)\zeta^{-c_r}(4s)M(s),$$

where $c_r = 2^{-r-1} + 2^{-2r-1} - 3^{-r} > 0$ and the Dirichlet series $M(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $\Re s > 1/5$.

Proof. Here $\tau^{(e)}(n)$ is multiplicative and by Euler product formula we have for $\sigma > 1$ that,

$$\begin{aligned}
\sum_{n=1}^{\infty}(\frac{\tau^{(e)}(n)}{n^s}) &= \prod_p (1 + \frac{(\tau^{(e)}(p))^{-r}}{p^s} + \frac{(\tau^{(e)}(p^2))^{-r}}{p^{2s}} + \frac{(\tau^{(e)}(p^3))^{-r}}{p^{3s}} + \cdots) \qquad (1)\\
&= \prod_p (1 + \frac{1}{p^s} + \frac{2^{-r}}{p^{2s}} + \frac{2^{-r}}{p^{3s}} + \frac{3^{-r}}{p^{4s}} \cdots)\\
&= \prod_p (1 - \frac{1}{p^s})^{-1} \prod_p (1 - \frac{1}{p^s})(1 + \frac{1}{p^s} + \frac{2^{-r}}{p^{2s}} + \frac{2^{-r}}{p^{3s}} \cdots)\\
&= \zeta(s)\zeta^{2^{-r}-1}(2s) \prod_p (1 - \frac{1}{p^s})^{2^{-r}-1}(1 + \frac{2^{-r}-1}{p^{2s}} + \frac{3^{-r}-2^{-r}}{p^{4s}} + \cdots)\\
&= \zeta(s)\zeta^{2^{-r}-1}(2s)\zeta^{-c_r}(4s)M(s).
\end{aligned}$$

So we get $c_r = 2^{-r-1} + 2^{-2r-1} - 3^{-r}$ and $M(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$. By the properties of Dirichlet series, the later one is absolutely convergent for $\Re es > 1/5$.

Lemma 2. Let $k \ge 2$ be a fixed integer, $1 < y \le x$ be large real numbers and

$$B(x, y; k, \epsilon) := \sum_{\substack{x < nm^k \le x+y \\ m > x^{\epsilon}}} 1.$$

Then we have

$$B(x, y; k, \epsilon) \ll yx^{-\epsilon} + x^{\frac{1}{2k+1}} \log x.$$

Proof. This is just a result of k-free number [4].

Let $a(n), b(n), c(n)$ be arithmetic functions defined by the following Dirichlet series (for $\Re s > 1$),

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \zeta(s)M(s), \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta^{2^{-r}-1}(s), \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \zeta^{-c_r}(s). \tag{4}$$

Lemma 3. Let $a(n)$ be an arithmetic function defined by (2), then we have

$$\sum_{n\le x} a(n) = A_1 x + O(x^{\frac{1}{5}+\epsilon}), \tag{5}$$

where $A_1 = \Re es_{s=1}\zeta(s)M(s)$.

Proof. Using Lemma 1, it is easy to see that

$$\sum_{n\le x} |g(n)| \ll x^{\frac{1}{5}+\epsilon}.$$

Therefore from the definition of $g(n)$ and (2), it follows that

$$\begin{aligned}\sum_{n\le x} a(n) &= \sum_{mn\le x} g(n) = \sum_{n\le x} g(n) \sum_{m\le \frac{x}{n}} 1 \\ &= \sum_{n\le x} g(n)(\frac{x}{n} + O(1)) = A_1 x + O(x^{\frac{1}{5}+\epsilon}),\end{aligned}$$

and $A_1 = \Re es_{s=1}\zeta(s)M(s)$.

Now we prove our Theorem. From Lemma 3 and the definition of $a(n)$, $b(n)$, $c(n)$, we get

$$(\tau^{(e)}(n))^{-r} = \sum_{n=n_1 n_2^2 n_3^4} a(n_1)b(n_2)c(n_3),$$

and

$$a(n) \ll n^{\epsilon^2}, b(n) \ll n^{\epsilon^2}, c(n) \ll n^{\epsilon^2}. \tag{6}$$

So we have

$$\begin{aligned}Q_r(x+y) - Q_r(x) &= \sum_{x<n_1 n_2^2 n_3^4 \le x+y} a(n_1)b(n_2)c(n_3) \\ &= \sum_1 + O(\sum_2 + \sum_3),\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\sum_1 &= \sum_{\substack{n_2 \le x^\epsilon \\ n_3 \le x^\epsilon}} b(n_2)c(n_3) \sum_{\frac{x}{n_2^2n_3^4} < n_1 \le \frac{x+y}{n_2^2n_3^4}} a(n_1), \\
\sum_2 &= \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_2 > x^\epsilon}} |a(n_1)b(n_2)c(n_3)|, \\
\sum_3 &= \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_3 > x^\epsilon}} |a(n_1)b(n_2)c(n_3)|.
\end{aligned} \tag{8}$$

By Lemma 3 we get

$$\begin{aligned}
\sum_1 &= \sum_{\substack{n_2 \le x^\epsilon \\ n_3 \le x^\epsilon}} b(n_2)c(n_3)\left(\frac{A_1 y}{n_2^2 n_3^4} + O\left(\left(\frac{x}{n_2^2 n_3^4}\right)^{\frac{1}{5}+\epsilon}\right)\right) \\
&= A_r y + O(yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}),
\end{aligned} \tag{9}$$

where $A_r = \Re es_{s=1} F(s)$. For Σ_2 we have by Lemma 2 and (6) that

$$\begin{aligned}
\sum_2 &\ll \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_2 > x^\epsilon}} (n_1 n_2 n_3)^{\epsilon^2} \\
&\ll x^{\epsilon^2} \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_2 > x^\epsilon}} 1 \\
&= x^{\epsilon^2} \sum_{\substack{x < n_1 n_2^2 \le x+y \\ n_2 > x^\epsilon}} d(1,4;n_1) \\
&\ll x^{2\epsilon^2} B(x,y;2,\epsilon) \\
&\ll x^{2\epsilon^2}(yx^{-\epsilon} + x^{\frac{1}{5}+\epsilon}) \\
&\ll yx^{2\epsilon^2-\epsilon} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}\log x \\
&\ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon},
\end{aligned} \tag{10}$$

if $\epsilon < 1/4$.

Similarly we have

$$\sum_3 \ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}. \tag{11}$$

Now our theorem follows from (7)-(11).

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On the mean value of some new sequences [1]

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Abstract The main purpose of this paper is to studied the mean value properties of some new sequences, and give several mean value formulae and their applications.
Keywords New sequences, mean value, asymptotic formula, applications.

§1. Introduction

For any monotonous increasing arithmetical function $g(n)$, we define two sequences $h(n)$ and $f(n)$ as follows: $h(n)$ is defined as the smallest positive integer k such that $g(k)$ greater than or equal to n. That is, $h(n) = \min\{k: \ g(k) \geq n\}$. $f(n)$ is defined as the largest positive integer k such that $g(k)$ less than or equal to n. That is, $f(n) = \max\{k: \ g(k) \leq n\}$. Further more, we let

$$S_n = (h(1) + h(2) + \cdots + h(n))/n;$$

$$I_n = (f(1) + f(2) + \cdots + f(n))/n;$$

$$K_n = \sqrt[n]{h(1) + h(2) + \cdots + h(n)};$$

$$L_n = \sqrt[n]{f(1) + f(2) + \cdots + f(n)}.$$

In references [1], Dr. Kenichiro Kashihara asked us to studied the properties of I_n, S_n, K_n and L_n. In references [3] and [4], Gou Su and Wang Yiren studied this problem, and obtained some interesting results. In this paper, we will use the elementary and analytic methods to study some similar problems, and prove a general result. As some applications of our theorem, we also give two interesting asymptotic formulae. That is, we shall prove the following:

Theorem. For any positive integer k, let $g(k) > 0$ be an increasing function, we have

$$S_n - I_n = \frac{1}{n}\left(\int_{M-1}^{M} g(t)\ dt + \int_{M-1}^{M} (t-[t])\, g'(t)\ dt + O(M)\right),$$

$$\frac{S_n}{I_n} = \frac{1}{n}\left(\frac{Mx - \int_0^{M-1} g(t)\ dt - \int_0^{M-1} (t-[t])\, g'(t)\ dt + O\left(M\right)}{Mx - \int_1^{M} g(t)\ dt - \int_1^{M} (t-[t])\, g'(t)\ dt + O\left(M\right)}\right),$$

and

$$\frac{K_n}{L_n} = \left(\frac{Mx - \int_0^{M-1} g(t)\ dt - \int_0^{M-1} (t-[t])\, g'(t)\ dt + O\left(M\right)}{Mx - \int_1^{M} g(t)\ dt - \int_1^{M} (t-[t])\, g'(t)\ dt + O\left(M\right)}\right)^{\frac{1}{n}}.$$

[1]This paper is supported by the N. S. F. of P.R.China.

As some applications of our theorem, we shall give two interesting examples. First we taking $g(k) = k^m$, that is $h(n) = \min\{k : \ k^m \geq n\}$ and $f(n) = \max\{k : \ k^m \leq n\}$. Then we have

Corollary 1. Let $g(k) = k^m$, then for any positive integer n, we have the asymptotic formulae

$$S_n - I_n = 1 + O\left(n^{-\frac{1}{m}}\right), \quad \frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{m}}\right),$$

and

$$\frac{K_n}{L_n} = 1 + O\left(\frac{1}{n}\right), \quad \lim_{n\to\infty} \frac{S_n}{I_n} = 1, \quad \lim_{n\to\infty} \frac{K_n}{L_n} = 1.$$

Next, we taking $g(k) = e^k$, that is $h(n) = \min\{k : \ e^k \geq n\}$ and $f(n) = \max\{k : \ e^k \leq n\}$, then we have:

Corollary 2. Let $g(k) = e^k$, then for any positive integer n, we have the asymptotic formulae

$$\frac{S_n}{I_n} = 1 + O\left(\frac{1}{\ln n}\right), \quad \frac{K_n}{L_n} = 1 + O\left(\frac{1}{n}\right),$$

$$\lim_{n\to\infty} \frac{S_n}{I_n} = 1, \quad \lim_{n\to\infty} \frac{K_n}{L_n} = 1.$$

§2. Proof of the theorem

In this section, we shall use the Euler summation formula and elementary method to complete the proof of our theorem. For any real number $x > 2$, it is clear that there exists one and only one positive integer M such that $g(M) < x < g(M+1)$. So we have

$$\begin{aligned}
\sum_{n\leq x} h(n) &= \sum_{k=1}^{M} \sum_{g(k-1)\leq n<g(k)} h(n) + \sum_{g(M)<n\leq x} h(n) \\
&= \sum_{g(0)\leq n<g(1)} h(n) + \sum_{g(1)\leq n<g(2)} h(n) + \cdots + \sum_{g(M)\leq n<x} h(n) \\
&= \sum_{k=1}^{M} k(g(k) - g(k-1)) + M\left(x - g(M)\right) + O\left(M\right) \\
&= Mg(M) - \sum_{k=0}^{M-1} g(k) + M\left(x - g(M)\right) + O\left(M\right) \\
&= Mx - \int_0^{M-1} g(t)\ dt - \int_0^{M-1} \left(t - [t]\right) g'(t)\ dt + O\left(M\right), \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n\le x} f(n) &= \sum_{k=1}^{M}\sum_{g(k)<n\le g(k+1)} f(n) - \sum_{x<n\le g(M+1)} f(n)\\
&= \sum_{g(1)<n\le g(2)} f(n) + \sum_{g(2)<n\le g(3)} f(n) + \cdots + \sum_{g(M)<n\le g(M+1)} f(n)\\
&\quad -M\left(g\left(M+1\right)-x\right)\\
&= \sum_{k=1}^{M} k(g(k+1)-g(k)) - M\left(g\left(M+1\right)-x\right)\\
&= (M+1)g(M+1) - \sum_{k=2}^{M+1} g(k) - M\left(g\left(M+1\right)-x\right)\\
&= Mx - \int_1^M g(t)\ dt - \int_1^M \left(t-[t]\right)g'(t)\ dt + O\left(M\right). \qquad (2)
\end{aligned}$$

In order to prove our theorem, we taking $x = n$ in (1) and (2), by using the elementary method we can get

$$\begin{aligned}
S_n - I_n &= \frac{1}{n}\left(h(1)+h(2)+\cdots+h(n)\right) - \frac{1}{n}\left(f(1)+f(2)+\cdots+f(n)\right)\\
&= \frac{1}{n}\left(\int_{M-1}^{M} g(t)\ dt + \int_{M-1}^{M}\left(t-[t]\right)g'(t)\ dt + O(M)\right). \qquad (3)
\end{aligned}$$

Then we have

$$\frac{S_n}{I_n} = \frac{Mx - \int_0^{M-1} g(t)\ dt - \int_0^{M-1}\left(t-[t]\right)g'(t)\ dt + O\left(M\right)}{Mx - \int_1^{M} g(t)\ dt - \int_1^{M}\left(t-[t]\right)g'(t)\ dt + O\left(M\right)},$$

and

$$\frac{K_n}{L_n} = \left(\frac{Mx - \int_0^{M-1} g(t)\ dt - \int_0^{M-1}\left(t-[t]\right)g'(t)\ dt + O\left(M\right)}{Mx - \int_1^{M} g(t)\ dt - \int_1^{M}\left(t-[t]\right)g'(t)\ dt + O\left(M\right)}\right)^{\frac{1}{n}}.$$

This completes the proof of our theorem.

Now we prove Corollary 1. Taking $g(k) = k^m$ in our theorem, for any real number $x > 2$, it is clear that there exists one and only one positive integer M satisfying $M^m < x < (M+1)^m$. That is, $M = x^{\frac{1}{m}} + O(1)$. So from our theorem we have

$$\begin{aligned}
\sum_{n\le x} h(n) &= \sum_{k=1}^{M}\sum_{(k-1)^m\le n<k^m} h(n) + \sum_{M^m<n\le x} h(n)\\
&= Mx - \int_0^{M-1} t^m\ dt - \int_0^{M-1}\left(t-[t]\right)\left(t^m\right)'\ dt + O\left(M\right)\\
&= Mx - \frac{1}{m+1}\left(M-1\right)^{m+1} + O\left(M^m\right).
\end{aligned}$$

Since $M = x^{\frac{1}{m}} + O(1)$, so we have the asymptotic formula

$$\sum_{n\le x} h(n) = \frac{m}{m+1}x^{\frac{m+1}{m}} + O(x).$$

Similarly, we have

$$\begin{aligned}\sum_{n\leq x} f(n) &= Mx - \int_1^M g(t)\ dt - \int_1^M (t-[t])\, g'(t)\ dt + O\left(M\right)\\ &= Mx - \frac{1}{m+1}\left(M-1\right)^{m+1} + O\left(M^m\right)\\ &= \frac{m}{m+1}x^{\frac{m+1}{m}} + O(x).\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}S_n - I_n &= \frac{1}{n}\left(\int_{M-1}^M g(t)\ dt + \int_{M-1}^M (t-[t])\, g'(t)\ dt + O(M)\right)\\ &= \frac{1}{n}M^m + O\left(\frac{M}{n}\right) = 1 + O\left(n^{-\frac{1}{m}}\right).\end{aligned}$$

$$\begin{aligned}\frac{S_n}{I_n} &= \frac{Mx - \int_0^{M-1} t^m\ dt - \int_0^{M-1}(t-[t])\,(t^m)'\ dt + O\left(M\right)}{Mx - \int_1^{M} t^m\ dt - \int_1^{M}(t-[t])\,(t^m)'\ dt + O\left(M\right)}\\ &= \frac{\frac{m}{m+1}n^{\frac{m+1}{m}} + O(n)}{\frac{m}{m+1}n^{\frac{m+1}{m}} + O(n)} = 1 + O\left(n^{-\frac{1}{m}}\right).\end{aligned}$$

$$\frac{K_n}{L_n} = \left(\frac{\frac{m}{m+1}n^{\frac{m+1}{m}} + O(n)}{\frac{m}{m+1}n^{\frac{m+1}{m}} + O(n)}\right)^{\frac{1}{n}} = 1 + O\left(\frac{1}{n}\right).$$

$$\lim_{n\to\infty}\frac{S_n}{I_n} = 1, \quad \lim_{n\to\infty}\frac{K_n}{L_n} = 1.$$

Now we prove Corollary 2. Taking $g(k) = e^k$ in our theorem. For any real number $x > 2$, it is clear that there exists one and only one positive integer M satisfying $e^M < x < e^{M+1}$, that is $M = \ln x + O(1)$. Then

$$\sum_{n\leq x} h(n) = Mx - \int_0^{M-1} e^t\ dt - \int_0^{M-1}(t-[t])\,(e^t)'\ dt + O\left(\ln x\right),$$

and

$$\sum_{n\leq x} f(n) = Mx - \int_1^{M} e^t\ dt - \int_1^{M}(t-[t])\,(e^t)'\ dt + O\left(\ln x\right).$$

Therefore,

$$\frac{S_n}{I_n} = \frac{Mx - \int_0^{M-1} e^t\ dt - \int_0^{M-1}(t-[t])\,(e^t)'\ dt + O\left(\ln x\right)}{Mx - \int_1^{M} e^t\ dt - \int_1^{M}(t-[t])\,(e^t)'\ dt + O\left(\ln x\right)} = 1 + O\left(\frac{1}{\ln n}\right),$$

$$\frac{K_n}{L_n} = \left(1 + O\left(\frac{1}{\ln n}\right)\right)^{\frac{1}{n}} = 1 + O\left(\frac{1}{n}\right),$$

and

$$\lim_{n\to\infty}\frac{S_n}{I_n} = 1, \quad \lim_{n\to\infty}\frac{K_n}{L_n} = 1.$$

This completes the proof of our corollaries.

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A short interval result for the exponential Möbius function

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Abstract The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i} > 1$ if $b_i | a_i$ for every $i \in \{1, 2, \cdots, s\}$. The exponential convolution of aithmetic functions is defined by

$$(f \bigodot g)(n) = \sum_{b_1 c_1 = a_1} \cdots \sum_{b_r c_r = a_r} = f(p_1^{b_1} \cdots p_r^{b_r}) g(p_1^{c_1} \cdots p_r^{c_r}),$$

where $n = \prod_{i=1}^{s} p_i^{a_i}$. The inverse of the constant function with respect to $\bigodot$ is called the exponential analogue of the Möbius function, which is denoted by $\mu^{(e)}$. The aim of this paper is to establish a short interval result for the function $\mu^{(e)}$.

Keywords The exponential divisor function, generalized divisor function, short interval.

§1. Introduction

Let $n > 1$ be an integer of canonical from $n = \prod_{i=1}^{s} p_i^{a_i}$. The integer $n = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of n if $b_i | a_i$ for every $i \in \{1, 2, \cdots, s\}$, notation: $d|_e n$. By convention $1|_e 1$.

Let $\mu^{(e)}(n) = \mu(a_1) \cdots \mu(a_r)$ here $n = \prod_{i=1}^{s} p_i^{a_i}$. Observe that $|\mu^{(e)}| = 0$ or $|\mu^{(e)}| = 1$, according as n is e-squarefree or not. The properties of the function $\mu^{(e)}$ is investigated by many authors. An asymptotic fomula for $A(x) := \sum_{n \leq x} \mu^{(e)}(n)$ was established by M. V. Subarao [2] and improved by J. Wu [1]. Recently László Tóth [3] proved that

$$A(x) = m(\mu^e) x + O(x^{\frac{1}{2}} \exp(-c (\log x)^{\Delta}), \tag{1}$$

where

$$m(\mu^e) := \prod_p (1 + \sum_{a=2}^{p} \frac{(\mu(a) - \mu(a-1))}{p^a}), \tag{2}$$

and $0 < \Delta < 9/25$ and $c > 0$ are fixed constants.

The aim of this paper is to study the short interval case and prove the following

Theorem. If $x^{\frac{1}{5} + \frac{3}{2}\epsilon} \leq y \leq x$, then

$$\sum_{x < n \leq x+y} \mu^{(e)}(n) = m(\mu^e) y + O(y x^{-\frac{1}{2}\epsilon} + x^{\frac{1}{5} + \frac{3}{2}\epsilon}),$$

where $m(\mu^e)$ is given by (2).

Notation. Throughout this paper, ϵ denotes a sufficiently small positive constant. $\mu(n)$ denotes the Möbius function. For fixed integers $1 \leq a \leq b$, the divisor function $d(a, b; n)$ is defined by

$$d(a, b; n) := \sum_{n=n_1^a n_2^b} 1.$$

§2. Proof of the theorem

Lemma 1. Suppose $\Re s > 1$, then we have

$$F(s) := \sum_{n=1}^{\infty} \frac{\mu^{(e)}(n)}{n^s} = \frac{\zeta(s)}{\zeta^2(2s)\zeta(5s)} G(s), \tag{1}$$

where the Dirichlet series $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for Re$s > 1/5$.

Proof. Since $\mu^{(e)}(n)$ is multiplicative, by Euler product formula we get for $\sigma > 1$ that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mu^{(e)}(n)}{n^s} &= \prod_p (1 + \frac{\mu(1)}{p^s} + \frac{\mu(2)}{p^{2s}} + \frac{\mu(3)}{p^{3s}} + \cdots) \\
&= \prod_p (1 - \frac{1}{p^s})^{-1} \prod_p (1 - \frac{1}{p^s})(1 + \frac{1}{p^s} - \frac{1}{p^{2s}} - \frac{1}{p^{3s}} + \cdots) \\
&= \zeta(s) \prod_p (1 - \frac{2}{p^{2s}} + \frac{1}{p^{4s}} - \frac{1}{p^{5s}} + \cdots) \\
&= \zeta(s) \prod_p (1 - \frac{2}{p^{2s}} + \frac{1}{p^{4s}} - \frac{1}{p^{5s}} + \cdots) \\
&= \frac{\zeta(s)}{\zeta^2(2s)\zeta(5s)} \prod_p (1 + \frac{2}{p^{6s} - p^s} - \frac{4}{p^{7s} - p^{2s}} + \frac{5}{p^{8s} - p^{3s}} + \cdots) \\
&= \frac{\zeta(s)}{\zeta^2(2s)\zeta(5s)} G(s),
\end{aligned}$$

where

$$G(s) = \prod_p (1 + \frac{2}{p^{6s} - p^s} - \frac{4}{p^{7s} - p^{2s}} + \frac{5}{p^{8s} - p^{3s}} + \cdots).$$

It is easily seen that $G(s)$ can be written as a Dirichlet series, which is absolutely convergent for $\Re s > 1/5$.

Lemma 2. Let $k \geq 2$ be a fixed integer, $1 < y \leq x$ be large real numbers and

$$B(x, y; k, \epsilon) : = \sum_{\substack{x < nm^k \leq x+y \\ m > x^{\epsilon}}} 1.$$

Then we have

$$B(x, y; k, \epsilon) \ll yx^{-\epsilon} + x^{\frac{1}{2k+1}} \log x.$$

Proof. See [4].

Let $a(n), b(n)$ be arithmetic functions defined by the following Dirichlet series (for $\Re s > 1$),

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \zeta(s)G(s), \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \frac{1}{\zeta^2(s)}. \tag{3}$$

Lemma 3. Let $a(n)$ be the arithmetic function defined by (2), then we have

$$\sum_{n\le x} a(n) = A_1 x + O(x^{\frac{1}{5}+\epsilon}), \tag{4}$$

where $A_1 = Res_{s=1}\zeta(s)G(s)$.

Proof. According to Lemma 1, it is easy to see that

$$\sum_{n\le x} |g(n)| \ll x^{\frac{1}{5}+\epsilon}.$$

Therefore from the definition of $a(n)$ and $g(n)$, it follows that

$$\begin{aligned}\sum_{n\le x} a(n) &= \sum_{mn\le x} g(n) = \sum_{n\le x} g(n) \sum_{m\le \frac{x}{n}} 1 \\ &= \sum_{n\le x} g(n)[\frac{x}{n}] = Ax + O(x^{\frac{1}{5}+\epsilon}).\end{aligned}$$

Now we prove our Theorem. From (1), the definitions of $a(n)$ and $b(n)$ we get that

$$\mu^{(e)}(n) = \sum_{n=n_1 n_2^2 n_3^5} a(n_1)b(n_2)\mu(n_3),$$

thus

$$\begin{aligned}A(x+y) - A(x) &= \sum_{x<n_1 n_2^2 n_3^5 \le x+y} a(n_1)b(n_2)\mu(n_3) \\ &= \sum_1 + O(\sum_2 + \sum_3),\end{aligned} \tag{5}$$

where

$$\begin{aligned}\sum_1 &= \sum_{\substack{n_2 \le x^\epsilon \\ n_3 \le x^\epsilon}} b(n_2)\mu(n_3) \sum_{\frac{x}{n_2^2 n_3^5} < n_1 \le \frac{x+y}{n_2^2 n_3^5}} a(n_1), \\ \sum_2 &= \sum_{\substack{x < n_1 n_2^2 n_3^5 \le x+y \\ n_2 > x^\epsilon}} |a(n_1)b(n_2)|, \\ \sum_3 &= \sum_{\substack{x < n_1 n_2^2 n_3^5 \le x+y \\ n_3 > x^\epsilon}} |a(n_1)b(n_2)|.\end{aligned}$$

By Lemma 3 we get easily that

$$\begin{aligned}\sum_1 &= \sum_{n_2\le x^\epsilon, n_3\le x^\epsilon} b(n_2)\mu(n_3)\left(\frac{A_1 y}{n_2^2 n_3^5} + O((\frac{x}{n_2^2 n_3^5})^{\frac{1}{5}+\epsilon})\right) \\ &= A_1 y \sum_{n_2\le x^\epsilon} b(n_2) n_2^{-2} \sum_{n_3\le x^\epsilon} \mu(n_3) n_3^{-5} + O(x^{\frac{1}{5}+\frac{3}{2}\epsilon})) \\ &= Ay + O(yx^{-\epsilon} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}),\end{aligned} \tag{6}$$

where $A = Res_{s=1}F(s) = m(\mu^e)$.

It is easily seen that the estimates

$$a(n) \ll n^{\epsilon^2},$$

and

$$b(n) = \sum_{n=n_1^2 n_2^2} \mu(n_1)\mu(n_2) \ll d(1,1;n) \ll n^{\epsilon^2},$$

hold, which combining the estimate $d(1,5;n) \ll n^{\epsilon^2}$ and Lemma 2 gives

$$\begin{aligned}\sum_2 &\ll \sum_{x<n_1 n_2^2 n_3^5 \le x+y} (n_1 n_2^2)^{\epsilon^2} \\ &\ll x^{\epsilon^2} \sum_{x<n_1 n_2^2 n_3^5 \le x+y} 1 \\ &\ll x^{2\epsilon^2} \sum_{x<n_1 n_2^2 \le x+y} d(1,5;n_1) \\ &\ll x^{2\epsilon^2}(yx^{-\epsilon} + x^{\frac{1}{5}+\epsilon}) \\ &\ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}.\end{aligned} \tag{7}$$

Similay we have

$$\sum_3 \ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}. \tag{8}$$

Now our theorem follows from (5)-(8).

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Approach of wavelet estimation in a semi-parametric regression model with ρ-mixing errors[1]

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Abstract Consider the wavelet estimator of a semi-parametric regression model with fixed design points when errors are a stationary ρ-mixing sequence. The wavelet estimators of unknown parameter and non-parameter are derived by the wavelet method. Under proper conditions, weak convergence rates of the estimators are obtained.

Keywords Semi-parametric regression model, wavelet estimator, ρ-mixing, convergence rate.

§1. Introduction

Consider a semi-parametric regression model

$$y_i = x_i\beta + g(t_i) + e_i,\ i = 1, 2, \cdots, n. \tag{1}$$

where $x_i \in R^1,\ t_i \in [0,1],\ \{(x_i, t_i),\ 1 \le i \le n\}$ is a deterministic design sequence, β is an unknown regression parameter, $g(\cdot)$ is an unknown Borel function, the unobserved process $\{e_i, 1 \le i \le n\}$ is a stationary ρ-mixing sequence and satisfies $Ee_i = 0,\ i = 1, 2, \cdots, n$.

In recent years, the parametric or non-parametric estimators in the semi-parametric or non-parametric regression model have been widely studied in the literature when errors are a stationary mixing sequence (example [1]-[4]). But up to now, the discussion of the wavelet estimators in the semi-parametric regression model, whose errors are a stationary ρ-mixing sequence, has been scarcely seen.

Wavelets techniques, due to their ability to adapt to local features of curves, have recently received much attention from mathematicians, engineers and statisticians. Many authors have applied wavelet procedures to estimate nonparametric and semi-parametric models ([5]-[7]).

In this article, we establish weak convergence rates of the wavelet estimators in the semi-parametric regression model with ρ-mixing errors, which enrich existing estimation theories and methods for semi-parametric regression models.

Writing wavelet scaling function $\phi(\cdot) \in S_q$ (q-order Schwartz space), multiscale analysis of

concomitant with $L^2(R)$ is V_m. And reproducing wavelet kernel of V_m is

$$E_m(t,s) = 2^m E_0(2^m t, 2^m s) = 2^m \sum_{k\in Z} \varphi(2^m t-k)\varphi(2^m s-k).$$

When β is known, we define the estimator of $g(\cdot)$,

$$\hat{g}_0(t,\beta) = \sum_{i=1}^{n}(y_i - x_i\beta)\int_{A_i} E_m(t,s)ds.$$

where $A_i = [s_{i-1}, s_i]$ is a partition of interval $[0,1]$ with $t_i \in A_i,\ 1\le i\le n$. Then, we solve the minimum problem $\min\limits_{\beta\in R}\sum\limits_{i=1}^{n}(y_i - x_i\beta - \hat{g}_0(t,\beta))^2$. Let its resolution be $\hat{\beta}_n$, we have

$$\hat{\beta}_n = \tilde{S}_n^{-2}\sum_{i=1}^{n}\tilde{x}_i\tilde{y}_i. \tag{2}$$

where $\tilde{x}_i = x_i - \sum\limits_{j=1}^{n} x_j\int_{A_j} E_m(t_i,s)ds,\ \tilde{y}_i = y_i - \sum\limits_{j=1}^{n} y_j\int_{A_j} E_m(t_i,s)ds,\ \tilde{S}_n^2 = \sum\limits_{i=1}^{n}\tilde{x}_i^2$.

Finally, we can define the estimator of $g(\cdot)$,

$$\hat{g}(t) \triangleq \hat{g}_0(t,\hat{\beta}_n) = \sum_{i=1}^{n}(y_i - x_i\hat{\beta}_n)\int_{A_i} E_m(t,s)ds. \tag{3}$$

§2. Assumption and lemmas

We assume that C and $C_i,\ i\ge 1$ express absolute constant, and they can express different values in different places.

2.1. Basic assumption

(A1) $g(\cdot)\in H^{\upsilon}(\upsilon>1/2)$ satisfies γ-order Lipschitz condition.

(A2) ϕ has compact support set and is a q-regular function.

(A3) $\left|\hat{\phi}(\xi)-1\right| = \phi(\xi),\ \xi\to 0$, where $\hat{\phi}$ is Fourier transformation of ϕ.

(A4) $\max\limits_{1\le i\le n}(s_i - s_{i-1}) = O(n^{-1}),\ \max\limits_{1\le i\le n}|\tilde{x}_i| = O(2^m)$.

(A5) $C_1\le \tilde{S}_n^2/n\le C_2$, where n is large enough.

(A6) $|\sum\limits_{i=1}^{n} x_i\int_{A_i} E_m(t,s)ds| = \lambda,\ t\in[0,1]$, where λ is a constant that depends only on t.

2.2. Lemmas

Lemma 1.[8] Let $\phi(\cdot)\in S_q$. Under basic assumptions of (A1)-(A3), if for each integer $k\ge 1$, $\exists c_k>0$, such that

(i) $|E_0(t,s)|\le \dfrac{c_k}{1+|t-s|^k},\ |E_m(t,s)|\le \dfrac{2^m c_k}{1+2^m|t-s|^k}$;

(ii) $\sup\limits_{t,m}\int_0^1 |E_m(t,s)|ds\le C$.

Lemma 2.[3] Under basic assumption (A1)-(A4), we have

(i) $\left|\int_{A_i} E_m(t,s)ds\right| = O(\frac{2^m}{n})$;

(ii) $\sum\limits_{i}^{n} (\int_{A_i} E_m(t,s)ds)^2 = O(\frac{2^m}{n})$.

Lemma 3.[3] Under basic assumption (A1)-(A5), we have

$$\sup_t \left|\sum_{i=1}^n g(t_i)\int_{A_i} E_m(t,s)ds - g(t)\right| = O(n^{-\gamma}) + O(\tau_m), n\to\infty.$$

where

$$\tau_m = \begin{cases} (2^m)^{-\alpha+1/2}, & (1/2<\alpha<3/2),\\ \sqrt{m}/2^m, & (\alpha=3/2),\\ 1/2^m, & (\alpha>3/2). \end{cases}$$

Lemma 4.[9] (Bernstein inequality) Let $\{x_i, i\in N\}$ be a ρ-mixing sequence and $E|x_i| = 0.$ $\{a_{ni},\ 1\le i\le n\}$ is a constant number sequence. $\{x_i\}\le d_i,\ a.s..$ $s_n = \sum\limits_{i=1}^n a_{ni}x_i$, then for $\forall\varepsilon>0$, such that

$$P(|s_n|>\varepsilon) \le C_1 \exp\left\{-t\varepsilon + 2C_2t^2\Delta + 2(l+1) + 2\ln(\rho(k))\right\}.$$

where constant C_1 do not depend on n, and $C_2 = 2(1+2\delta)(1+8\sum\limits_{i=1}^{\infty}\rho(i)),\ \Delta = \sum\limits_{i=1}^{\infty} a_{ni}^2 d_i^2,\ t>0.$ l and k satisfy the following inequality

$$2lk \le n \le 2(l+1)k,\ tk\cdot \max_{1\le i\le n}|a_{ni}d_i| \le \delta \le \frac{1}{2}.$$

§3. Main results and proofs

Theorem 1. Under basic assumption (A1)-(A6), if $\sum\limits_{i=1}^{\infty}\rho(i)<\infty,\ \Lambda_n = \max(n^{-\gamma},\tau_m)$, and if $\exists\ d=d(n)\in N$, such that $d\Lambda_n\to\infty,\ \dfrac{n}{d^2\Lambda_n}\to 0$ and $\dfrac{2^m d^3}{n\Lambda_n}\to 0$,
then, we have

$$\hat\beta_n - \beta = O_p(\Lambda_n), \tag{4}$$

$$\hat g(t) - g(t) = O_p(\Lambda_n). \tag{5}$$

Proof. It is obvious that

$$\begin{aligned}\hat\beta_n - \beta &= \tilde S_n^{-2}(\sum_{i=1}^n \tilde x_i\tilde e_i + \sum_{i=1}^n \tilde x_i\tilde g_i)\\ &= \tilde S_n^{-2}[\sum_{i=1}^n \tilde x_i e_i - \sum_{i=1}^n \tilde x_i \sum_{j=1}^n e_j\int_{A_j} E_m(t_i,s)ds + \sum_{i=1}^n \tilde x_i\tilde g_i]\\ &\triangleq B_{1n} + B_{2n} + B_{3n},\end{aligned} \tag{6}$$

where $\tilde e_i = e_i - \sum\limits_{j=1}^n e_j\int_{A_j} E_m(t_i,s)ds,\ \tilde g_i = g_i - \sum\limits_{j=1}^n g_j\int_{A_j} E_m(t_i,s)ds.$

Since $|B_{1n}| = \left|\sum\limits_{i=1}^n \tilde S_n^{-2}\tilde x_i e_i\right| \triangleq \left|\sum\limits_{i=1}^n b_i e_i\right|.$

Hence

$$\begin{aligned} P(|B_{1n}|>\eta\Lambda_n) &= P(\Big|\sum_{i=1}^{n} b_i e_i\Big|>\eta\Lambda_n) \\ &\le P(\Big|\sum_{i=1}^{n} b_i e_i I(|e_i|\le d)\Big|>\eta\Lambda_n)+P(\Big|\sum_{i=1}^{n} b_i e_i I(|e_i|>d)\Big|>\eta\Lambda_n) \\ &\triangleq P(J_{n1})+P(J_{n2}), \end{aligned} \tag{7}$$

where $d=d(n)\in N$.

By assumption (A4) and (A5), we have

$$|b_i|=\left|\tilde{S}_n^{-2}\tilde{x}_i\right|=O\left(\frac{2^m}{n}\right),\ \sum_{i=1}^{n} b_i^2=\sum_{i=1}^{n}(\tilde{S}_n^{-2}\tilde{x}_i)^2 b_i^2\le \max_i \tilde{S}_n^{-2}\tilde{x}_i\cdot\sum_{i=1}^{n}\tilde{S}_n^{-2}\tilde{x}_i=O\left(\frac{2^m}{n}\right).$$

If further $d=t=k$ in Lemma 4, then the conditions of Lemma 4 suffice.

By applying Markov's inequality and the conditions of Theorem 1, we have

$$P(J_{n2})\le\frac{E(\Big|\sum_{i=1}^{n} b_i e_i I(|e_i|>d)\Big|)}{\eta\Lambda_n}\le\frac{Ee_i^2\cdot\sum_{i=1}^{n} b_i}{d\eta\Lambda_n}=\frac{Ee_i^2\cdot\sum_{i=1}^{n}(n^{-1}\tilde{x}_i)\cdot(n\tilde{s}_n^{-2})}{d\eta\Lambda_n}\le\frac{C}{d\Lambda_n}\to 0. \tag{8}$$

By Lemma 4, we have

$$\begin{aligned} P(J_{n1}) &\le C_1\exp\left\{-t\Lambda_n\eta+C_2t^2\Delta+2(l+1)+2\ln(\rho(k))\right\} \\ &\le C_1\exp\left\{-d\Lambda_n\eta+C_2d^2\Delta+2(l+1)\right\}. \end{aligned}$$

Since

$$\Delta=\sum_{i=1}^{n} b_i^2 d^2=\frac{2^m d^2}{n},$$

So

$$\frac{d^2\Delta}{d\Lambda_n}=\frac{2^m d^3}{n\Lambda_n}\to 0,\quad \frac{l+1}{d\Lambda_n}\le\frac{n/2k+1}{d\Lambda_n}=\frac{n+2d}{2d^2\Lambda_n}=\frac{n}{2d^2\Lambda_n}+\frac{1}{d\Lambda_n}\to 0.$$

Therefore $\dfrac{(l+1)+d^2\Delta}{d\Lambda_n}\to 0.$

Hence

$$P(J_{n1})\le C_1\cdot\exp\left\{-\frac{d}{2}\Lambda_n\eta\right\}\to 0\qquad(d\to\infty). \tag{9}$$

Thus, we have shown

$$B_{1n}\to o_p(\Lambda_n). \tag{10}$$

Write $b_j=\sum_{i=1}^{n}\tilde{S}_n^{-2}\tilde{x}_i\int_{A_j}E_m(t_i,s)ds$.

Then $|B_{2n}|=|\sum_{j=1}^{n} b_j e_j|$.

Note that

$$|b_j|\le\sum_{i=1}^{n}\tilde{S}_n^{-2}\tilde{x}_i\cdot\max_i\int_{A_j}E_m(t_i,s)ds=n\tilde{S}_n^{-2}\cdot n^{-1}\sum_{i=1}^{n}\tilde{x}_i\cdot\max_i\int_{A_j}E_m(t_i,s)ds=O(\frac{2^m}{n}),$$

$$\sum_{j=1}^{n} b_j^2=\sum_{j=1}^{n}(\sum_{i=1}^{n}\tilde{S}_n^{-2}\tilde{x}_i\int_{A_j}E_m(t_i,s)ds)^2\le\sum_{i=1}^{n}(\tilde{S}_n^{-2}\tilde{x}_i)^2\cdot\max_i\sum_{j=1}^{n}\int_{A_j}E_m(t_i,s)ds\le C\cdot\frac{2^{2m}}{n^2}.$$

Similarly to the proof of B_{1n}, we obtain

$$B_{2n} \to o_p(\Lambda_n). \tag{11}$$

Consider now B_{3n}. By assumption (A6) and Lemma 3, we have

$$|B_{3n}| \le \tilde{S}_n^{-2} \sum_{i=1}^{n} |\tilde{x}_i| \cdot \max_{1\le i\le n} |\tilde{g}_i| = O(n^{-\gamma}) + O(\tau_m). \tag{12}$$

Thus, by (6), (10), (11) and (12), the proof of (4) is completed.
It is easy to see the following decompositions:

$$\begin{aligned} \sup_t |\hat{g}(t) - g(t)| &\le \sup_t |\hat{g}_0(t,\beta) - g(t)| + \sup_t \left| \sum_{i=1}^n x_i(\beta - \hat{\beta}_n) \int_{A_i} E_m(t,s)ds \right| \\ &\le \sup_t \left| \sum_{i=1}^n g(t_i) \int_{A_i} E_m(t,s)ds - g(t) \right| + \sup_t \left| \sum_{i=1}^n e_i \int_{A_i} E_m(t,s)ds \right| \\ &\quad + \left| \beta - \hat{\beta}_n \right| \cdot \sup_t \left| \sum_{i=1}^n x_i \int_{A_i} E_m(t,s)ds \right| \triangleq T_1 + T_2 + T_3. \end{aligned} \tag{13}$$

By Lemma 2, it is clear that

$$T_1 = O(n^{-\gamma}) + O(\tau_m). \tag{14}$$

Analogous to the proof of (4), and note that (13), (14) and (A6), it is clear that

$$\hat{g}(t) - g(t) = O_p(\Lambda_n).$$

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Generalize derivations on semiprime rings

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Abstract The purpose of this paper is to study the concept of generalize derivations in the sense of Nakajima on semi prime rings, we proved that a generalize Jordan derivation (f, ∂) on a ring R is a generalize derivation if R is a commutative or a non commutative 2-torsion free semi prime with ∂ is symmetric hochschilds 2-cocycles of R. Also we give a necessary and sufficient condition for generalize derivations (f, ∂_1), (g, ∂_2) on a semi prime ring to be orthogonal.

Keywords Prime ring, semi prime ring, generalize derivation, derivation, orthogonal derivation.

§1. Introduction

Throughout R will represent an associative ring with $Z(R)$. R is said to be 2-torsion free if $2x = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use the basic commutator identites $[xy, z] = [x, z]y + x[y, z]$ and is $[x, yz] = y[x, y] + [x, y]z$. Recall that a ring R is prime if $aRb = 0$ implies that either $a = 0$ or $b = 0$, and R is semi prime if $aRa = 0$ implies $a = 0$. An additive mapping d: $R \to R$ is called derivation if $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$. And d is called Jordan derivation if $d(a^2) = d(a)a + ad(a)$ for all $a \in R$. In [3], M. Bresar introduced the definition of generalize derivation on rings as follows: An additive map g: $R \to R$ is called a generalize derivation if there exists a derivation d: $R \to R$ such that $g(ab) = g(a)b + ad(b)$ for all $a, b \in R$ (we will call it of type 1). It is clear that every derivation is generalize derivation, and an additive map J: $R \to R$ is called Jordan generalize if there exists a derivation d: $R \to R$ such that $J(a^2) = J(a)a + ad(a)$ for all $a \in R$ (we will call it of type 1). An additive map ∂: $R \times R \to R$ be called hochschild 2-cocylc if $x(y, z) - \partial(xy, z) + \partial(x, yz) - \partial(x, y)z = 0$ for all $x, y, z \in R$, the map ∂ is called symmetric if $\partial(x, y) = \partial(y, x)$ for all $x, y \in R$. It is clear that every Jordan derivation is Jordan generalize derivation, and every generalize derivation is Jordan generalize derivation, and since Jordan derivation may be not derivation, so Jordan generalize derivation, may be not generalize derivation in general. The properties of this type of mapping were discussed in many papers ([1],[2],[7]), especially, in [2], M. Ashraf and N. Rehman showed that if R is a 2-torsion free ring which has a commtator non zero divisor, then every Jordan generalize derivation on R is

generalize derivation. In [7], Nurcan Argac, Atsushi Nakajima and Emine Albas introduced the notion of orthogonality for a pair f, g of generalize derivation, and they gave several necessary and sufficient conditions for f, g to be orthogonal. Nakajima [1] introduced another definition of generalize derivation as following: An additive mapping f: $R \to R$ is called generalize derivation if there exist a 2 -coycle ∂: $R\times R \to R$ such that $f(ab) = f(a)b+af(b)+\partial(a,b)$ for all $a, b \in R$ (we will call it of type 2). And is called Jordan generalize derivation if $f(a^2) = f(a)a+af(a)+\partial(a,a)$ for all $a \in R$ (we will call it of type 2). In this paper we work on the generalize derivation (of type 2), we will extend the result of ([1], Theorem 1) and also we give several sufficient and necessary conditions which makes two generalize derivation (of type 2) to be orthogonal. For a ring R, let U be a subset of R, then the left annihilator of U (rep, right annihilator of U) is the set $a \in R$ such that $aU = 0$ (res, is the set $a \in R$ such that $Ua = 0$). We denote the annihilator of U by $Ann(U)$. Note that $U \cap Ann(U) = 0$ and $U \oplus Ann(U)$ is essential ideal of R.

§2. Preliminaries and examples

In this section, we give some examples and some well-known lemmas we are needed in our work.

Example 2.1. Let R be a ring $d : R \to R$ be a derivation and $\partial : R \times R \to R$ defined by $\partial(a, b) = 2d(a)d(b)$ and $g : R \to R$ is defined as follows $g(a) = d(d(a))$ for all $a \in R$. $g(a + b) = d(d(a + b)) = d(d(a) + d(b)) = d(d(a)) + d(d(b)) = g(a) + g(b)$ for all $a, b \in R$.

And, $g(ab) = d(d(ab)) = d(d(a)b + ad(b)) = d(d(a)b) + d(ad(b)) = d(d(a))b + d(a)d(b) + d(a)d(b) + ad(d(b)) = g(a)b + ag(b) + 2d(a)d(b) = g(a)b + ag(b) + \partial(a, b)$ for all $a, b \in R$. Hence, g is generalize derivation (of type 2) on R .

The following Example explains that the definition of generalize derivation (of type 2) is more generalizing than the generalize derivation (of type 1).

Example 2.2. Let (f, d) be a generalize derivation on a ring R then the map (f, ∂) is generalize derivation, where $\partial(a, b) = a(d - f)(b)$ for all $a, b \in R$.

Lemma 2.1.[5] Let R be a 2-torsion free ring, $(f, \partial) : R \to R$ be generalize Jordan derivation, then $f(ab + ba) = f(ab) + f(ba) = f(a)b + af(b) + \partial(a, b) + f(b)a + bf(a) + \partial(b, a)$ for all $a, b \in R$.

Lemma 2.2.[5] Let R be a 2-torsion free ring and $(f, \partial) : R \to R$ be generalize Jordan derivation the map S: $R \times R \to R$ defined as follows: $S(a, b) = f(ab) - (f(a)b + af(b) + \partial(a, b)$ for all $a, b \in R$. And the map $[\ \ ] : R \times R \to R$ defined by $[a, b] = ab - ba$ for all $a, b \in R$, then the following relations hold

(1) $S(a, b)c[a, b] + [a, b]cS(a, b) = 0$ for all $a, b \in R$.

(2) $S(a, b)[a, b] = 0$ for all $a, b \in R$.

Lemma 2.3. Let R be a 2-torsion free ring, $(f, \partial) : R \to R$ be generalize Jordan derivation the map $S : R \times R \to R$ defined as follows: $S(a, b) = f(ab) - (f(a)b + af(b) + \partial(a, b))$ for all $a, b \in R$, then the following relations hold

(1) $S(a_1 + a_2, b)$=$S(a_1, b) + S(a_2, b)$ for all $a_1, a_2, b \in R$.

(2) $S(a, b_1 + b_2)$=$S(a, b_1) + S(a, b_2)$ for all $a, b_1, b_2 \in R$.

Proof.

$$\begin{aligned}
(1)\ S(a_1+a_2,b) &= f((a_1+a_2)b)-(f(a_1+a_2)b+(a_1+a_2)f(b)+(a_1+a_2,b))\\
&= f(a_1b)+f(a_2b)-(f(a_1)b+f(a_2)b+a_1f(b)+a_2f(b)\\
&\quad +\partial(a_1,b)+\partial(a_2,b))\\
&= f(a_1b)-(f(a_1)b+a_1f(b)+\partial(a_1,b))+f(a_2b)-(f(a_2)b\\
&\quad +a_2f(b)+\partial(a_2,b))\\
&= S(a_1,b)+S(a_2,b).
\end{aligned}$$

$$\begin{aligned}
(2)\ S(a,b_1+b_2) &= f(a(b_1+b_2))-(f(a)(b_1+b_2)+af(b_1+b_2)+\partial(a,b_1+b_2))\\
&= f(ab_1)+f(ab_2)-(f(a))b_1+f(a)b_2+af(b_1)+af(b_2)+\\
&\quad \partial(a,b_1)+\partial(a,b_2)\\
&= f(ab_1)-(f(a))b_1+af(b_1)+\partial(a,b_1)+f(ab_2)-f(a)b_2\\
&\quad +af(b_2)+\partial(a,b_2)\\
&= S(a,b_1)+S(a,b_2).
\end{aligned}$$

Lemma 2.4.[1] If R is a 2-torsion free semi prime ring and a, b are elements in R then the following are equivalent.

(i) $a\times b=0$ for all x in R.

(ii) $b\times a=0$ for all x in R.

(iii) $a\times b+b\times a=0$ for all x in R. If one of them fulfilled, then $ab=ba=0$.

§3. Generalize Jordan derivations on a semi prime rings

In this section, we extend the result proved by Nakajima ([1], Theorem 1(1), (2))by adding condition.

Theorem 3.1. If R is a commutative 2-torsion free ring and $(f,\partial):R\to R$ be a generalize Jordan derivation then (f,∂) is a generalize derivation.

Proof. Let $S(a,b)=f(ab)-f(a)b-af(b)-\partial(a,b)$ for all $a,b\in R$. And by Lemma (2.1) we have $S(a,b)+S(b,a)=0$. So

$$S(a,b)=-S(b,a). \tag{1}$$

And, $f(ab-ba)=f(a)b+af(b)+\partial(a,b)-f(b)a-bf(a)-\partial((b,a)$. Since R is commutative. Then

$$S(a,b)=f(ab)-f(a)b-af(b)-\partial(a,b)=f(ba)-bf(a)-f(b)a-\partial(b,a)\ =S(b,a). \tag{2}$$

From (1) and (2), we get $2S(a,b)=0$. And since R is 2-torsion free ring, so $S(a,b)=0$. Hence f is a generalize derivation.

Theorem 3.2. Let R be a non commutative 2-torsion free semi prime ring and (f,∂) : $R\to R$ be a generalize Jordan derivation then (f,∂) is a generalize where ∂ is symmetric.

Proof. Let $a, b \in R$, then by Lemma (2.2,(1)) $S(a,b)c[a,b] + [a,b]cS(a,b) = 0$, for all $c \in R$. And since R is semi prime and by Lemma (2.4), we get $S(a,b)c[a,b] = 0$ for all $c \in R$, it is clear that S are additive maps on each argument, then:

$$S(a,b)c[x,y] = 0 \text{ for all } x, y, c \in R. \tag{1}$$

Now,

$$\begin{aligned} 2S(x,y)wS(x,y) &= S(x,y)w(S(x,y) + S(x,y))S(x,y)w(S(x,y) - S(y,x)) \\ &= S(x,y)w(f(x,y) - (f(x)y + xf(y) + \partial(x,y)) \\ &\quad -(f(yx) - (f(y)x - yf(x) - \partial(y,x)) \end{aligned}$$

Since ∂ is symmetric, so $\partial(x,y) = \partial(y,x)$, then

$$\begin{aligned} 2S(x,y)wS(x,y) &= S(x,y)w(f(x,y) - f(x)y - xf(y) - f(yx) + f(y)x + yf(x)) \\ &= S(x,y)w((f(xy) - f(yx) + [f(y),x] + [y,f(x)]) \\ &= S(x,y)w(f(xy - yx) + [f(y),x] + [y,f(x)]) \\ &= S(x,y)wf(xy - yx) + S(x,y)w[f(y),x] + S(x,y)w[y,f(x)]. \end{aligned}$$

By equation (1), we get $2S(x,y)wS(x,y) = S(x,y)wf(xy - yx) = S(x,y)w(f(x)y + xf(y) + \partial(x,y) - f(y)x - yf(x) - \partial(y,x))$ Since ∂ is symmetric. Thus $2S(x,y)wS(x,y) = S(x,y)w([f(x),y] + [x,f(y)]) = S(x,y)w[f(x),y] + S(x,y)w[x,f(y)]$. Then by equation (1), we get $2S(x,y)wS(x,y) = 0$ for all $x, y, w \in\in R$, and since R is 2-torsion free, so $S(x,y)wS(x,y) = 0$ for all $x, y, w \in R$, since R is prime ring Thus, $S(x,y) = 0$ for all $x, y \in R$, then $f(xy) = f(x)y + xf(y) + \partial(x,y)$. Hence, f is generalize derivation on R.

§4. Orthogonal generalize derivations on semi prime rings

In this section, we gave some necessary and sufficient conditions for tow generalized derivation of type (2), to be orthogonal and also we show that the image of two orthogonal generalize derivations are different from each other except for both are is zero.

Definition 4.1. Two map f and g on a ring R are orthogonal if $f(x)Rg(y) = 0$ for all $x, y \in R$.

Lemma 4.1. If R is a ring $\partial : R \to R$ is a hochschild 2-cocycle, $S = R \oplus R$, and $\partial : S \times S \to S$ defined by $\partial : ((x_1, y_1), (x_2, y_2)) = (\partial(x_1, x_2), 0)$ for all $(x_1, y_1), (x_2, y_2) \in S \times S$ is a hochschild 2-cocycle.

Proof. It is clear that ∂ is an additive map in each argument and $((x_1, x_2)\partial((y_1, y_2), (z_1, z_2))$ $-\partial((x_1, x_2)(y_1, y_2), (z_1, z_2)) + \partial((x_1, x_2), (y_1, y_2)(z_1, z_2)) - \partial((x_1, x_2), (y_1, y_2)(z_1, z_2)) = (x_1\partial(y_1, z_1)$ $-\partial(x_1y_1, z_1) + \partial(x_1, y_1z_1) - \partial(x_1, y_1)z_1, 0) = (0, 0)$ for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in S \times S$. Thus, ∂ is a hochschild 2-cocycle.

Theorem 4.1. For any generalize derivation (f, ∂_1) on a ring R there exist tow orthogonal generalize derivation (h, ∂_1), (g, ∂_2) on $S = R \oplus R$ such that $h(x,y) = (f(x), 0)$, $g(x,y) = (0, f(y))$ for all $x, y \in R$.

Proof. Let $h : S \to S$ and $g : S \to S$ defined as following $h(x,y) = (f(x), 0)$ for all $(x,y) \in S$, $g(x,y) = (0, f(y))$ for all $(x,y) \in S$, and let ∂_1 defined as in Lemma 4.1. Now it is clear that

h, g are additive mapping and $h((x, y)(z, w)) = h(xz, yz) = (f(xz), 0) =(f(x)z, 0)+(xf(z), 0)+(\partial(x, z), 0) =(f(x), 0)(z, w)+(x, y)(f(z), 0)+(\partial(x, z), 0) =h(x, y)(z, w)+(x, y)h(z, w)+\partial((x, y)(z, w))$. Thus (h, ∂) is a generalize derivation on S. In similar way (g, ∂) is a generalize derivation on S, and $h(x, y)(m, n)g(z, w) = (f(x), 0)(m, n)(0, f(w)) = (0, 0)$ for all $(x, y), (m, n), (z, w) \in S$. $h(x, y)Sg(z, w) = 0$ for all $(x, y), (z, w) \in S$. Hence h, g are orthogonal generalize derivation on S.

Theorem 4.2. If $(f, \partial_1), (g, \partial_2)$ are a generalize derivation on commutative semi prime ring R with identity, then the following are equivalent:

(i) (f, ∂_1) and (g, ∂_2) are orthogonal.

(ii) $f(x)g(y) = 0$ and $\partial_1(x, y)g(w) = 0$ for all $x, y, w \in R$.

(iii) $f(x)g(y) = 0$ and $\partial_2(x, y)f(w) = 0$ for all $x, y, w \in R$.

(iv) There exist tow ideal U, V of R such that

(a) $U \cap V = 0$.

(b) $f(R) \subseteq U$ and $g(R), \partial_2(R, R) \subseteq V$.

Proof. $(i) \to (ii)$ Since $f(x)Rg(y) = 0$ for all $x, y \in R$, then by Lemma 2.4, we get $f(x)g(y) = 0$ for all $x, y \in R$. Now $0 = f(xy)g(w) = f(x)yg(w) + xf(y)g(w) + \partial_1(x, y)g(w)$. But $f(x)yg(w) = 0 = xf(y)g(w)$, so $\partial_1(x, y)g(w) = 0$ for all $x, y, w \in R$.

$(i) \to (iii)$ In similar way of $(i) \to (ii)$.

$(i) \to (iv)$ Let U be an ideal of R generated by $f(R)$ and $V = Ann(U)$, then by Lemma 4.1, $U \cap V = 0$ and by (ii). $\partial_2(x, y)f(w) = 0$ for all $x, y, w \in R$, and since R is commutative ring with identity, so it is clear that $U =$ where $r_i \in R, s_i \in f(R)$ and n is any positive integer=0, thus clear that $\partial_2(x, y)U = 0$ and since $f(x)g(y) = 0$ for all $x, y \in R$, then where $r_i \in R, s_i \in f(R)$ and n is any positive integer. $g(y) = 0$, for all $y \in R$. $Ug(y) = 0$ for all $y \in R$, then $\partial_2(R, R), g(R) \subseteq Ann(U) = V$.

$(ii) \to (i)$ For all $x, y, w \in R$, $f(xy)g(w) = 0 = f(x)yg(w)+xf(y)g(w)+\partial_1(x, y)g(w)$. But $f(y)g(w) = 0 = \partial_1(x, y)g(w)$. So, $f(x)yg(w) = 0$, $f(x)Rg(w) = 0$. Thus f, g are orthogonal generalize derivations.

$(iii) \to (i)$ In similar way of $(ii) \to (i)$.

$(iv) \to (i)$ Since $f(R) \subseteq U$, $f(R)g(R) = 0$, then $g(R) \subseteq V$, Thus, $f(R)Rg(R) \subseteq U \cap V$, then $f(R)Rg(R) = 0$. Hence f, g are orthogonal generalize derivations. This completes the proof.

Corollary 4.1. If $(f, \partial_1), (g, \partial_2)$ are a generalize derivation on commutative semi prime ring R with unity, then $f(R) \cap g(R) = 0$.

Proof. By Theorem 4.2 there exists tow ideals U, V of R such that $U \cap V = 0$ And $f(R) \subseteq U$ and $g(R) \subseteq V$. $f(R)g(R) \subseteq U \cap V$, hence $f(R) \cap g(R) = 0$.

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Near meanness on product graphs

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Abstract Let $G = (V, E)$ be a graph with p vertices and q edges and let $f : V(G) \to \{0, 1, 2, \cdots, q-1, q+1\}$ be an injection. The graph G is said to have a near mean labeling if for each edge, there exist an induced injective map $f^* : E(G) \to \{1, 2, \cdots, q\}$ defined by

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2}, & \text{if } f(u)+f(v) \text{ is even,} \\ \dfrac{f(u)+f(v)+1}{2}, & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

The graph that admits a near mean labeling is called a near mean graph (NMG). In this paper, we proved that the graphs Book B_n, Ladder L_n, Grid $P_n \times P_n$, Prism $P_m \times C_3$ and $L_n \odot K_1$ are near mean graphs.

Keywords Near mean labeling, near mean graph.

§1. Introduction

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph G denoted are by $V(G)$ and $E(G)$ respectively. The Cartesian product of graph $G_1(V_1, E_1) \& G_2(V_2, E_2)$ is $G_1 \times G_2$ and is defined to be a graph whose vertex set is $V_1 \times V_2$ and edge set is $\{\{(u_1, v_1), (u_2, v_2)\}$: either $u_1 = u_2$ and $v_1 v_2 \in E_2$ or $v_1 = v_2$ and $u_1 u_2 \in E_1\}$. The graphs $K_{1,n} \times K_2$ is book $B_n, P_n \times K_2$ is ladder $L_n, P_n \times P_n$ is grid $L_{n,n}, P_m \times C_3$ is prism and $L_n \odot K_1$ is corono of ladder. Terms and notations not used here are as in [2].

§2. Preliminaries

The mean labeling was introduced in [3]. Let G be a (p, q) graph and we define the concept of near mean labeling as follows.

Let $f : V(G) \to \{0, 1, 2, \cdots, q-1, q+1\}$ be an injection and also for each edge $e = uv$ it induces a map $f^* : E(G) \to \{1, 2, \cdots, q\}$ defined by

$$f^*(uv) = \begin{cases} \dfrac{f(u)+f(v)}{2}, & \text{if } f(u)+f(v) \text{ is even,} \\ \dfrac{f(u)+f(v)+1}{2}, & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

A graph that admits a near mean labeling is called a near mean graph. We have proved in [4], $P_n, C_n, K_{2,n}$ are near mean graphs and K_n $(n > 4)$ and $K_{1,n}$ $(n > 4)$ are not near mean graphs. In [5], we proved family of trees, Bi-star, Sub-division Bi-star $P_m \odot 2K_1, P_m \odot 3K_1$, $P_m \odot K_{1,4}$ and $P_m \odot K_{1,3}$ are near mean graphs. In this paper, we proved that the graphs Book B_n, Ladder L_n, Grid $P_n \times P_n$, Prism $P_m \times C_3$ and $L_n \odot K_1$ are near mean graphs.

§3. Near meanness on product graphs

Theorem 3.1. Book $K_{1,n} \times K_2$ (n-even) is a near mean graph.

Proof. Let $K_{1,n} \times K_2 = \{V, E\}$ such that

$$V = \{(u, v, u_i, v_i : 1 \leq i \leq n)\},$$
$$E = \{[(uu_i) \cup (vv_i) : 1 \leq i \leq n] \cup (uv) \cup [(u_iv_i) : 1 \leq i \leq n]\}.$$

We define $f : V \to \{0, 1, 2, \cdots, 3n, 3n + 2\}$ by

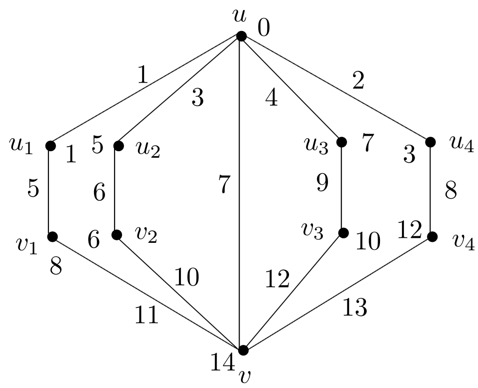
$$\begin{aligned}
f(u) &= 0,\\
f(v) &= 3n + 2,\\
f(u_i) &= 4i - 3,\ 1 \leq i \leq \frac{n}{2},\\
f(u_{n+1-i}) &= 4i - 1,\ 1 \leq i \leq \frac{n}{2},\\
f(v_i) &= 2n - 2(i - 1),\ 1 \leq i \leq \frac{n}{2},\\
f(v_{n+1-i}) &= 3n - 2(i - 1),\ 1 \leq i \leq \frac{n}{2}.
\end{aligned}$$

The induced edge labelings are

$$\begin{aligned}
f^*(uu_i) &= 2i - 1,\ 1 \leq i \leq \frac{n}{2},\\
f^*(uu_{n+1-i}) &= 2i,\ 1 \leq i \leq \frac{n}{2},\\
f^*(u_iv_i) &= n + i,\ 1 \leq i \leq \frac{n}{2},\\
f^*(uv) &= \frac{3n + 2}{2},\\
f^*(u_{n+1-i}v_{n+1-i}) &= \frac{3n + 2}{2} + i,\ 1 \leq i \leq \frac{n}{2},\\
f^*(vv_i) &= \frac{5n}{2} + 2 - i,\ 1 \leq i \leq \frac{n}{2},\\
f^*(vv_{n+1-i}) &= 3n + 2 - i,\ 1 \leq i \leq \frac{n}{2}.
\end{aligned}$$

It can be easily seen that each edge gets different label from $\{1, 2, \cdots, 3n+1\}$. Hence $K_{1,n} \times K_2$ (n is even) is a near mean graph.

Example 3.2.



$K_{1,4} \times K_2$

Theorem 3.3. Book $K_{1,n} \times K_2$ (n-odd) is a near mean graph.

Proof. Let $K_{1,n} \times K_2 = \{V, E\}$ such that

$$V = \{(u, v, u_i, v_i : 1 \leq i \leq n)\},$$
$$E = \{[(uu_i) \cup (vv_i) : 1 \leq i \leq n] \cup (uv) \cup [(u_iv_i) : 1 \leq i \leq n]\}.$$

We define $f : V \to \{0, 1, 2, \cdots, 3n, 3n + 2\}$ by

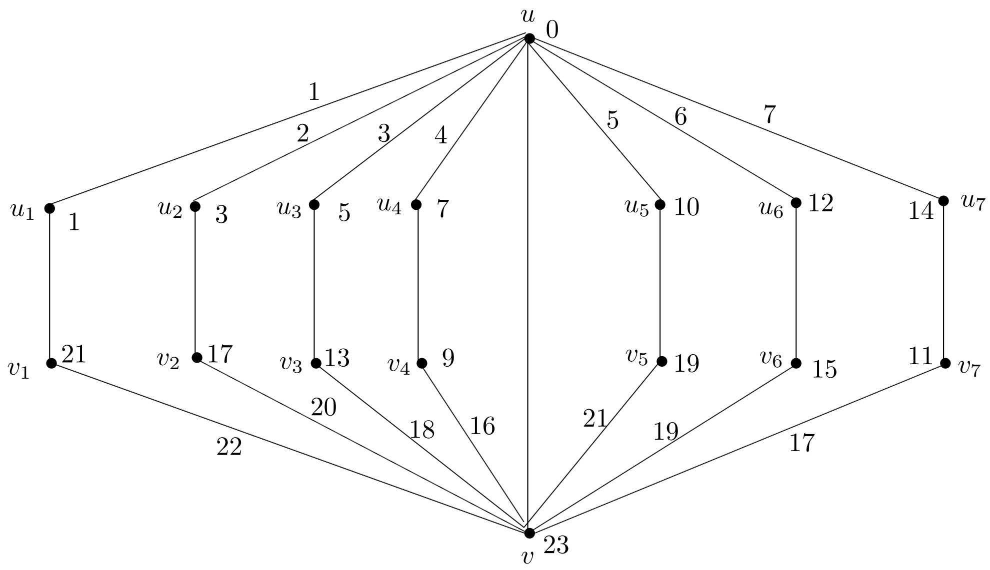
$$f(u) = 0,$$
$$f(v) = 3n + 2, \text{ Let } x = \frac{n+1}{2},$$
$$f(u_i) = \begin{cases} 2i - 1, & \text{if } 1 \leq i \leq x, \\ 2i, & \text{if } x + 1 \leq i \leq n, \end{cases}$$
$$f(v_i) = 3n - 4(i - 1), \ 1 \leq i \leq x,$$
$$f(v_{x+i}) = 3n - 2 - 4(i - 1), \ 1 \leq i \leq n - x.$$

The induced edge labelings are

$$f^*(uu_i) = i, \ 1 \leq i \leq n,$$
$$f^*(u_iv_i) = \frac{3n+3}{2} - i, \ 1 \leq i \leq x,$$
$$f^*(uv) = \frac{3n+3}{2},$$
$$f^*(u_{n+1-i}v_{n+1-i}) = \frac{3n+3}{2} + i, \ 1 \leq i \leq x - 1,$$
$$f^*(vv_i) = 3n + 3 - 2i, \ 1 \leq i \leq x,$$
$$f^*(vv_{x+i}) = 3n + 2 - 2i, \ 1 \leq i \leq x - 1.$$

It can be easily seen that each edge gets different label from $\{1, 2, \cdots, 3n+1\}$. Hence, $K_{1,n} \times K_2$ (n is odd) is a near mean graph.

Example 3.4.



$K_{1,7} \times K_2$

Theorem 3.5. Ladder $L_n = P_n \times K_2$ is a near mean graph.

Proof. Let $V(P_n \times K_2) = \{u_i, v_i : 1 \le i \le n\}$.

$E(P_n \times K_2) = \{[(u_iu_{i+1}) \cup (v_iv_{i+1}) : 1 \le i \le n-1] \cup [(u_iv_i) : 1 \le i \le n]\}$.

Define $f : V(P_n \times K_2) \to \{0, 1, 2, \cdots, 3n-3, 3n-1\}$ by

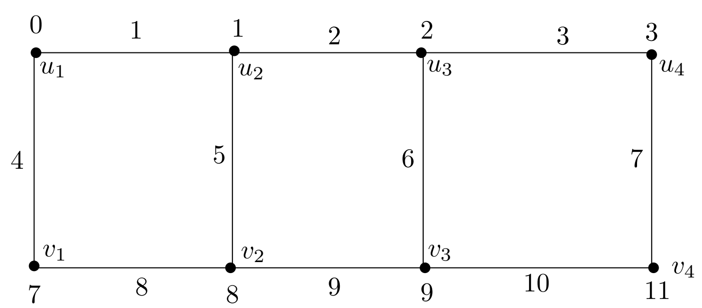
$$f(u_i) = i - 1,\ 1 \le i \le n,$$
$$f(v_i) = 2n - 2 + i,\ 1 \le i \le n-1,$$
$$f(v_n) = 3n - 1.$$

The induced edge labelings are

$$f^*(u_iu_{i+1}) = i,\ 1 \le i \le n-1,$$
$$f^*(v_iv_{i+1}) = 2n + i - 1,\ 1 \le i \le n-1,$$
$$f^*(u_iv_i) = n + i - 1,\ 1 \le i \le n.$$

Hence, $P_n \times K_2$ is a near mean graph.

Example 3.6.



$P_4 \times K_2$

Theorem 3.7. The Grid graph $P_n \times P_n$ admits near mean labeling.

Proof. Let $V(P_n \times P_n) = \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq n\}$,

$E(P_n \times P_n) = \{[(u_{ij}u_{ij+1}) : 1 \leq i \leq n, 1 \leq j \leq n-1] \cup [(u_{ij}u_{i+1j}) : 1 \leq i \leq n-1, 1 \leq j \leq n]\}$.

We define $f : V \to \{0, 1, 2, \cdots, q-1, q+1\}$ by

For $i = 1, 2, 3, \cdots, n-1$,

$$f(u_{ij}) = (i-1)(2n-1) + (j-1),\ 1 \leq j \leq n.$$

For $i = n$,

$$f(u_{nj}) = (n-1)(2n-1) + (j-1),\ 1 \leq j \leq n-1,$$
$$f(u_{nn}) = 2n(n-1) + 1.$$

The induced edge labelings are

For $i = 1, 2, 3, \cdots, n$,

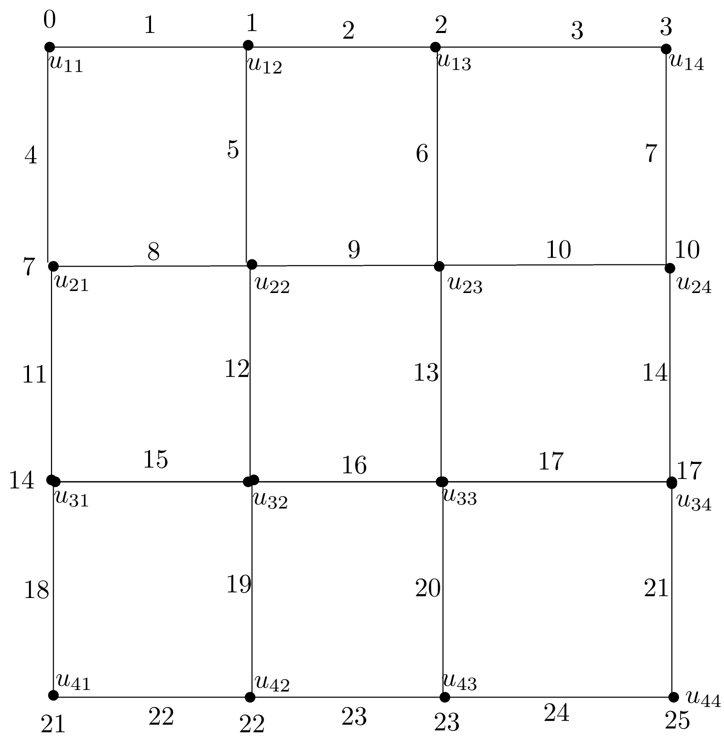
$$f^*(u_{ij}, u_{i,j+1}) = (i-1)(2n-1) + j,\ 1 \leq j \leq n-1.$$

For $j = 1, 2, 3, \cdots, n$,

$$f^*(u_{ij}, u_{i+1,j}) = (i-1)(2n-1) + (j-1) + n,\ 1 \leq i \leq n-1.$$

It can be easily verifty that each edge gets different label from the set $\{1, 2, \cdots, q\}$. Hence, $P_n \times P_n$ is near a mean graph.

Example 3.8.



$P_4 \times P_4$

Theorem 3.9. Prism $P_m \times C_3$ $(m \geq 2)$ is a near mean graph.

Proof. Let $P_m \times C_3 = G(V, E)$ such that

$$\begin{aligned} &V = \{u_i, v_i, w_i : 1 \leq i \leq m\}, \\ &E = \{[u_i u_{i+1}) \cup (v_i v_{i+1}) \cup (w_i w_{i+1}) : 1 \leq i \leq m-1] \cup \\ &\quad [(u_i v_i) \cup (v_i w_i) \cup (u_i w_i) : 1 \leq i \leq m]\}. \end{aligned}$$

We define $f : V \to \{0, 1, 2, \cdots, 6m-4, 6m-2\}$ by

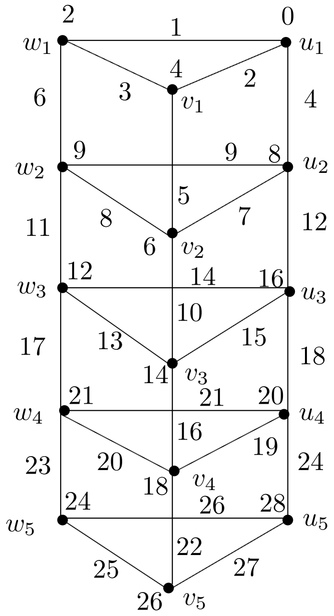
$$\begin{aligned} f(u_1) &= 0, \\ f(v_1) &= 4, \\ f(w_1) &= 2, \\ f(w_m) &= 6m-2, \text{ if } m \text{ is even}, \\ f(u_i) &= \begin{cases} 6i-4, & i = 0 \bmod 2,\ 2 \leq i \leq m \\ 6i-2, & i = 1 \bmod 2,\ 3 \leq i \leq m, \end{cases} \\ f(v_i) &= \begin{cases} 6i-4, & i = 1 \bmod 2,\ 3 \leq i \leq m, \\ 6i-6, & i = 0 \bmod 2,\ 2 \leq i \leq m, \end{cases} \\ f(w_i) &= \begin{cases} 6i-3, & i = 0 \bmod 2,\ 2 \leq i \leq m, \\ 6i-6, & i = 1 \bmod 2,\ 3 \leq i \leq m. \end{cases} \end{aligned}$$

The induced edge labelings are

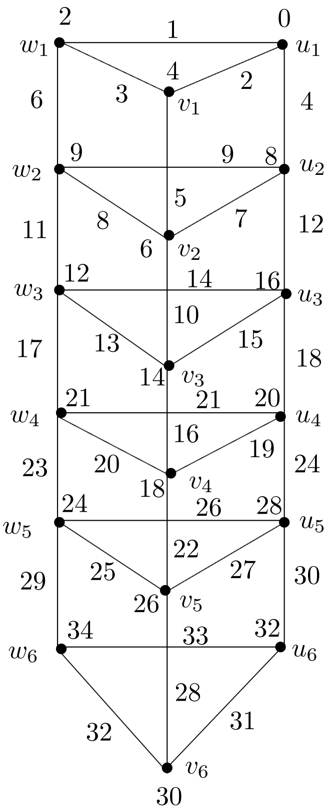
$$\begin{aligned} f^*(u_1 v_1) &= 2, \\ f^*(u_1 u_2) &= 4, \\ f^*(v_1 w_1) &= 3, \\ f^*(v_1 v_2) &= 5, \\ f^*(u_1 w_1) &= 1, \\ f^* w_1 w_2) &= 6, \\ f^*(u_i v_i) &= \begin{cases} 6i-5, & i = 0 \bmod 2,\ 2 \leq i \leq m, \\ 6i-3, & i = 1 \bmod 2,\ 3 \leq i \leq m, \end{cases} \\ f^*(v_i w_i) &= \begin{cases} 6i-4, & i = 0 \bmod 2,\ 2 \leq i \leq m, \\ 6i-5, & i = 1 \bmod 2,\ 3 \leq i \leq m, \end{cases} \\ f^*(u_i w_i) &= \begin{cases} 6i-3, & i = 0 \bmod 2,\ 2 \leq i \leq m, \\ 6i-4, & i = 1 \bmod 2,\ 3 \leq i \leq m, \end{cases} \\ f^*(u_i u_{i+1}) &= 6i,\ 2 \leq i \leq m-1, \\ f^*(v_i v_{i+1}) &= 6i-2,\ 2 \leq i \leq m-1, \\ f^*(w_i w_{i+1}) &= 6i-1,\ 2 \leq i \leq m-1. \end{aligned}$$

It is clear that each edge gets unique labeling from the set $\{1, 2, 3, \cdots, 6m-3\}$. Hence, Prism $P_m \times C_3$ is a near mean graph.

Example 3.10.



$P_5 \times C_3$: (m-odd)



$P_6 \times C_3$: (m-even)

Theorem 3.11. The graph $L_n \odot K_1 = (P_2 \times P_n) \odot K_1$ admits near mean labeling.

Proof. Let $L_n \odot K_1 = \{V, E\}$ such that

$$V = \{u_i, v_i : 1 \leq i \leq 2n\},$$
$$E = \{[u_i u_{i+1}) : 1 \leq i \leq 2n-1] \cup (u_1 u_{2n}) \cup [(u_i v_i) : 1 \leq i \leq 2n]$$
$$\cup [u_i u_{2n+1-i}) : 2 \leq i \leq n-1]\}.$$

We define $f : V \to \{0, 1, 2, \cdots, 5n-3, 5n-1\}$ by

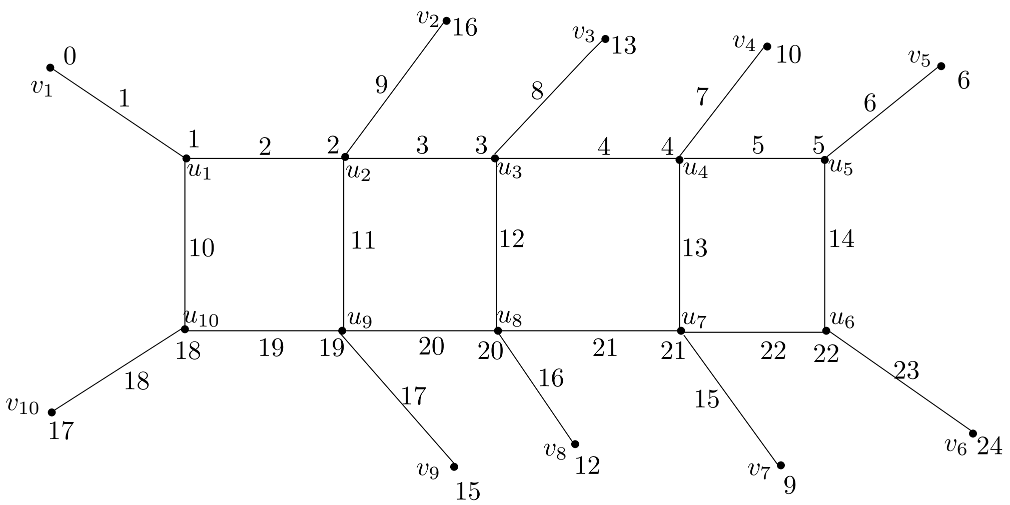
$$f(u_i) = i, \ 1 \leq i \leq n,$$
$$f(u_{n+i}) = 5n - 2 - i, \ 1 \leq i \leq n,$$
$$f(v_1) = 0,$$
$$f(v_n) = n + 1,$$
$$f(v_{n+1}) = 5n - 1,$$
$$f(v_{2n}) = 4n - 3,$$
$$f(v_{n-i}) = n + 2 + 3i, \ 1 \leq i \leq n-2,$$
$$f(v_{n+1+i}) = n + 1 + 3i, \ 1 \leq i \leq n-2.$$

The induced edge labelings are

$$\begin{aligned} f^*(u_1v_1) &= 1, \\ f^*(u_nv_n) &= n+1, \\ f^*(u_{n+1}v_{n+1}) &= 5n-2, \\ f^*(v_{2n}u_{2n}) &= 4n-2, \\ f^*(u_iu_{i+1}) &= i+1,\ 1 \le i \le n-1, \\ f^*(u_nu_{n+1}) &= 3n-1, \\ f^*(u_{2n}u_1) &= 2n, \\ f^*(u_{n+i}u_{n+i+1}) &= 5n-2-i,\ 1 \le i \le n-1, \\ f^*(u_iv_i) &= 2n+1-i,\ 2 \le i \le n-1, \\ f^*(u_{n+1+i}v_{n+1+i}) &= 3n+(i-1),\ 1 \le i \le n-2, \\ f^*(u_iu_{2n+1-i}) &= 2n+(i-1),\ 2 \le i \le n-1. \end{aligned}$$

Clearly edges get distinguished labels from $\{1, 2, \cdots, q\}$. Hence $L_n \odot K_1$ is a near mean graph.

Example 3.12. $L_n \odot K_1 = (P_2 \times P_5) \odot K_1$.



Theorem 3.13. Cuboid $C_4 \times P_m (m > 2)$ is a mean graph.

Proof. Let $C_4 \times P_m = G(V, E)$ such that

$$\begin{aligned} V(G) &= \{u_{ij} : 1 \le i \le m, 1 \le j \le 4\}, \\ E(G) &= \{[u_{ij}u_{ij+1}) : 1 \le i \le m, 1 \le j \le 3] \cup [(u_{i4}u_{i1}) : 1 \le i \le m] \cup \\ &\quad [(u_{ij}u_{i+1j}) : 1 \le i \le m-1, 1 \le j \le 4]\}. \end{aligned}$$

We define $f : V(G) \to \{0, 1, 2, \cdots, q-1, q+1\}$ by

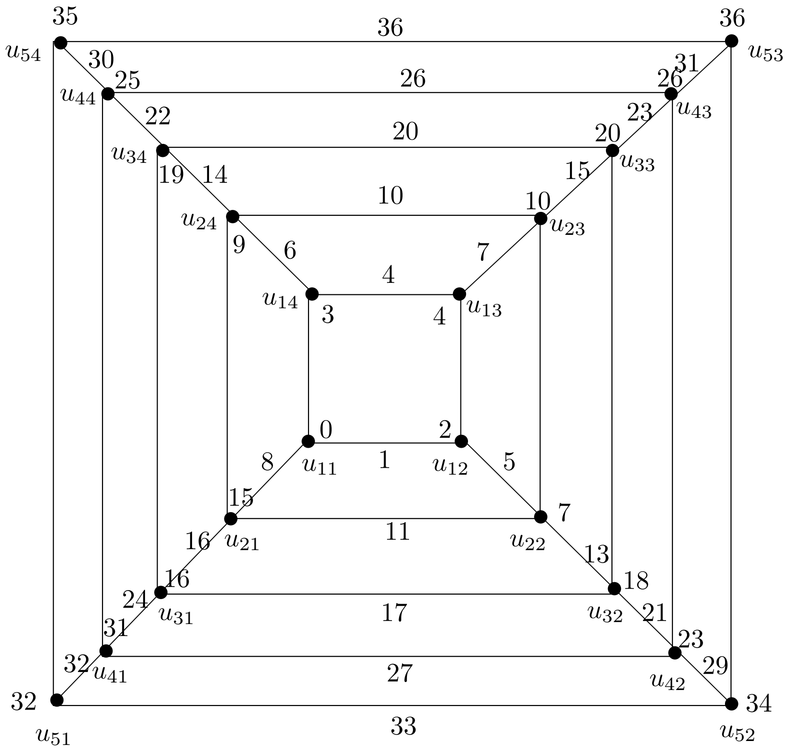
$$f(u_{11}) = 0,$$
$$f(u_{12}) = 2,$$
$$f(u_{13}) = 4,$$
$$f(u_{14}) = 3,$$
$$f(u_{i1}) = \begin{cases} f(u_{i-1}) + 15, & i = 0 \bmod 2,\ 2 \le i \le m, \\ f(u_{i-1}) + 1, & i = 1 \bmod 2,\ 3 \le i \le m, \end{cases}$$
$$f(u_{i2}) = \begin{cases} f(u_{i-1}) + 5, & i = 0 \bmod 2,\ 2 \le i \le m, \\ f(u_{i-1}) + 11, & i = 1 \bmod 2,\ 3 \le i \le m, \end{cases}$$
$$f(u_{i3}) = \begin{cases} f(u_{i-1}) + 6, & i = 0 \bmod 2,\ 2 \le i \le m, \\ f(u_{i-1}) + 10, & i = 1 \bmod 2,\ 3 \le i \le m, \end{cases}$$
$$f(u_{i4}) = \begin{cases} f(u_{i-1}) + 6, & i = 0 \bmod 2,\ 2 \le i \le m, \\ f(u_{i-1}) + 10, & i = 1 \bmod 2,\ 3 \le i \le m, \end{cases}$$

The induced edge labelings are

$$f^*(u_{11}u_{12}) = 1,$$
$$f^*(u_{12}u_{13}) = 3,$$
$$f^*(u_{13}u_{14}) = 4,$$
$$f^*(u_{14}u_{11}) = 2,$$
$$f^*(u_{i1}u_{i2}) = \begin{cases} f(u_{i-11}u_{i-12}) + 10, & i = 0 \bmod 2,\ 2 \le i \le m, \\ f(u_{i-11}u_{i-12}) + 6, & i = 1 \bmod 2,\ 3 \le i \le m, \end{cases}$$
$$f^*(u_{i2}u_{i3}) = \begin{cases} f(u_{i-12}u_{i-13}) + 6, & i = 0 \bmod 2,\ 2 \le i \le m, \\ f(u_{i-12}u_{i-13}) + 10, & i = 1 \bmod 2,\ 3 \le i \le m, \end{cases}$$
$$f^*(u_{i3}u_{i4}) = \begin{cases} f(u_{i-13}u_{i-14}) + 6, & i = 0 \bmod 2,\ 2 \le i \le m, \\ f(u_{i-13}u_{i-14}) + 10, & i = 1 \bmod 2,\ 3 \le i \le m, \end{cases}$$
$$f^*(u_{i4}u_{i1}) = \begin{cases} f(u_{i-14}u_{i-11}) + 10, & i = 0 \bmod 2,\ 2 \le i \le m, \\ f(u_{i-14}u_{i-11}) + 6, & i = 1 \bmod 2,\ 3 \le i \le m, \end{cases}$$
$$f^*(u_{i1}u_{i+11}) = 8i,\ 1 \le i \le m-1,$$
$$f^*(u_{i2}u_{i+12}) = 8i - 3,\ 1 \le i \le m-1,$$
$$f^*(u_{i3}u_{i+13}) = 8i - 1,\ 1 \le i \le m-1,$$
$$f^*(u_{i4}u_{i+14}) = 8i - 2,\ 1 \le i \le m-1.$$

It is clear that each edge gets distinct labeling from the set $\{1, 2, \cdots, q\}$. Hence $C_4 \times P_m (m > 2)$ is a near mean graph.

Example 3.14.



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Super-self-conformal sets

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Abstract In this paper we will give two different definitions of super-self-conformal sets. About the first definition, some properties of the dimensions are obtained. About the second definition, we will prove that the super-self-conformal sets and their generating IFS are not defined each other only.

Keywords Super-self-conformal set, Hausdorff dimension, open set condition.

§1. Introduction

Self-similar sets presented by Hutchinson [4] have been extensively studied. See for example [3], [5], [8], [9]. Let's recall that. Let $X \subseteq R^n$ be a nonempty compact convex set, and there exists $0 < C < 1$ such that

$$|w(x) - w(y)| \le C\,|x - y|\,, \quad \forall x, y \in X.$$

Then we say that $w : X \to X$ is a *contractive map.* If each w_i $(1 \le i \le m)$ is a contractive map from X to X, then we call $(X, \{w_i\}_{i=1}^m)$ the contractive iterated function systems (IFS). It is proved by Hutchinson that if $(X, \{w_i\}_{i=1}^m)$ is a contractive IFS, then there exists a unique nonempty compact set $E \subset R^n$, such that

$$E = \bigcup_{i=1}^{m} w_i(E). \tag{1}$$

Set E is called an attractor of IFS$\{w_i\}_{i=1}^m$. If each w_i is a contractive self-similar map, then we call $(X, \{w_i\}_{i=1}^m)$ the contractive self-similar IFS. Set E in (1) is an attractor of $\{w_i\}_{i=1}^m$, and it is called *self-similar set.* If each w_i is a contractive self-conformal map, then $(X, \{w_i\}_{i=1}^m)$ is called the contractive self-conformal IFS, and set E decided by (1) is called *self-conformal set*, see [1], [2], [7].

Recently, Falconer introduced sub-self-similar sets and super-self-similar sets. Let $\{w_i\}_{i=1}^m$ is self-similar IFS, and let F be a nonempty compact subset of R^n such that

$$F \subseteq \bigcup_{i=1}^{m} w_i(F).$$

This set F is called *sub-self-similar set* for $\{w_i\}_{i=1}^m$ [6]. Here if A such that $A \supseteq \bigcup_{i=1}^{m} w_i(A)$, and A is a nonempty subset of R^n, then we call A a super-self-similar set. Easy to see that,

self-similar sets is a class of special sub-self-similar sets. At the same time, Falconer obtained the formula for the Hausdorff and box dimension of the sub-self-similar sets, if $\{w_i\}_{i=1}^m$ satisfies the open set condition (OSC). In [11], we gave the definition of sub-self-conformal sets similarly:

Suppose $(X, \{w_i\}_{i=1}^m)$ is a contractive self-conformal IFS, and a nonempty compact subset of X, F satisfies the condition

$$F \subseteq \bigcup_{i=1}^m w_i(F).$$

Then we call F a *sub-self-conformal set* for$\{w_i\}_{i=1}^m$. And we obtained the formula for their Hausdorff and box dimension.

In this paper, we will give two definitions of super-self-conformal sets. At the same time we will discuss their different properties respectively.

We organize this paper as follows. In section 2, we will give the first definition:

Definition 1. Let $(X, \{w_i\}_{i=1}^m)$ is a contractive self-conformal IFS with attractor E. If A is a non-empty compact subset of E satisfying

$$A \supseteq \bigcup_{i=1}^m w_i(A),$$

then we call A a *super-self-conformal set*.

At the same time, we obtain the property about the dimensions of super-self-conformal sets under this definition. Next, we give the second definition:

Definition 2. Let $(X, \{w_i\}_{i=1}^m)$ be contracting conformal IFS. If A is a non-empty compact set satisfying

$$A \supseteq \bigcup_{i=1}^m w_i(A),$$

then we call A a *super-self-conformal set*.

Here we will consider the relations of the super-self-conformal sets and its generating IFS.

Our main results are:

Theorem 1. Let $(X, \{w_i\}_{i=1}^m)$ be contracting conformal set with attractor E. If A is a non-empty compact subset of E satisfying

$$A \supseteq \bigcup_{i=1}^m w_i(A).$$

And $\dim_H A = s$ then $\underline{\dim}_B E = \overline{\dim}_B E = s$ and $H^s(A) < \infty$.

Theorem 2. Let $(X, \{w_i\}_{i=1}^m)$ be a contractive conformal IFS. Then the super-self-conformal set generated by $\{w_i\}_{i=1}^m$ is uncountable. Conversely, There exists a super-self-conformal set A which its generating IFS is uncountable. Moreover, the attractor for each IFS is different.

§2. Proofs of the main results

Here, we discuss the super-self-conformal sets defined by Definition 1 first. We will get the property of the dimensions.

Lemma 1.[7] Let E be a non-empty compact subset of R^n and let $a > 0$ and $r_0 > 0$. Suppose that for every closed ball B with centre in$E$ and radius $r < r_0$ there is a mapping $g : E \to E \cap B$ satisfying

$$ar\,|x - y| \le |g(x) - g(y)|\,, \qquad (x, y \in E).$$

Then, writing$s = \dim_H E$, we have that $H^s(E) \le 4^s a^{-s} < \infty$ and $\underline{\dim}_B E = \overline{\dim}_B E = s$.

Lemma 2. Let $(X, \{w_i\}_{i=1}^m)$ be a self-conformal contractive IFS with $\{|w_i'(x)|\}_{i=1}^m$ satisfying

$$0 < \inf_x |w_i'(x)| \le \sup_x |w_i'(x)| < 1, \text{ for each } 1 \le i \le m,$$

and OSC. Let E be the self-conformal set for $(X, \{w_i\}_{i=1}^m)$. Then $\dim_H(E) = \underline{\dim}_B(E) = \overline{\dim}_B(E) = s$, here s satisfying $\tau(s) = \lim\limits_{k\to\infty} \left(\sum\limits_I |w_I'(x)|^s\right)^{\frac{1}{k}} = 1$. Moreover, $0 < H^s(E) < \infty$.

The Theorem 1.1 in [1] and Theorem 2.7 in [7] give the results.

Theorem 1. Let $(X, \{w_i\}_{i=1}^m)$ be contracting conformal set with attractor E. If A is a non-empty compact subset of E satisfying

$$A \supseteq \bigcup_{i=1}^m w_i(A).$$

And $\dim_H A = s$ then $\underline{\dim}_B E = \overline{\dim}_B E = s$ and $H^s(A) < \infty$.

Proof. If $x \in A \subseteq E$, and r is small enough. Write $r_{\min} = \min\limits_{x, 1\le i\le m} |w_i'(x)|$, there exist a sequence $(i_1, i_2, \cdots, i_k)$ such that $x \in w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_k}(A)$ for all k. Choose k so

$$r_{\min} \cdot r < \left|w_{i_1}'(x)\right| \cdot \left|w_{i_2}'(x)\right| \cdots \left|w_{i_k}'(x)\right| \cdot |A| \le r.$$

Then $w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_k} : A \to A \cap B(x, r)$ is a ratio at least $r_{\min} |A|^{-1} r$ so Lemma 1 and Lemma 2 give equality of the dimensions and $H^s(A) < \infty$.

Here we must point out that by the hypothesis $A \subseteq E$, we have $A = E$. In fact,

$$E = \bigcap_{k=1}^\infty w^k(A) \subseteq w(A) = \bigcup_{i=1}^m w_i(A) \subseteq A.$$

So we get rid of the condition $A \subseteq E$, and lead to the second definition.

In this part, we are interesting in this question: Are the super-self-conformal and its generating IFS defined each other only? Theorem 2 will give the answer.

Let E be an attractor of a contractive conformal IFS $(X, \{w_i\}_{i=1}^m)$. We write

$$A_k = \bigcup_{n=1}^k \bigcup_{|I|} w_I(X),$$

$$A_0 = \lim_{k\to\infty} A_k,$$

$$A(X) = A_0 \cup E.$$

Lemma 3. A_0 is exist, and $A(X)$ is a compact set.

Proof. For A_k are monotone increasing sets, and X is a compact subset of R^n, and $\{w_i\}_{i=1}^m$ is a contractive conformal IFS, so $A_0 = \lim\limits_{k\to\infty} A_k$ is exist. Moreover,

$$A_0 = \lim_{k\to\infty} A_k = \bigcup_{k=1}^{\infty} A_k,$$

then A_0 is a bounded set. To prove the compactness of $A(X)$, we just need to prove that $A(X)$ is close.

Let $x_i \in A(X), (i = 1, 2, \cdots)$, then $\{x_i\} \subset E$ or $\{x_i\} \subset A_0$. (We may choose sub sequence of $\{x_i\}$ for necessary.)

When $\{x_i\} \subset E$. E is compact, so $x_0 \in E$.

When $\{x_i\} \subset A_0$. For any $i \geq 1$, we have $x_i \in A_0$. Then there exist an unique $l = l(i)$ such that $x_i \in A_k \backslash A_{k-1}$. We write

$$l_0 = \max\{l(i) : i \geq 1\}.$$

If $l_0 < \infty$, then $x_i \in A_{l_0}$ for any i. We know that $x_0 \in A(X)$, for A_{l_0} is compact.
If $l_0 = \infty$, then there exist $y_i \in X$ and $I_i \in \Sigma$ such that

$$\lim_{i\to\infty} w_{I_i}(y_i) = x_0.$$

It shows that $x_0 \in E$. So $x_0 \in A(X)$.

From [10], we can find a similar proof.

Lemma 4. There exists a super-self-conformal set A which generating IFS is uncountable. Moreover, the attractor for each IFS is different.

Proof. Consider the following IFS on $A = [0, 1]$:

$$w_1(x) = \frac{1}{2}x^2, \quad w_2(x) = rx^2 + (1 - r).$$

Here $r \in \left(0, \frac{1}{3}\right]$. Obviously,

$$A \supset w_1(A) \cup w_2(A).$$

A is a super-self-conformal set generated by these IFS. It has uncountable selection for r, so the generating IFS of A is uncountable. Clearly, every choice such that IFS satisfies OSC. For each IFS, the Hausdorff dimension of the attractor is s, here s satisfies $\tau(s) = 1$. It means s change with r. Therefore each IFS has different attractor.

Theorem 2. Let $(X, \{w_i\}_{i=1}^m)$ be a contractive conformal IFS. Then the super-self-conformal set generating by $\{w_i\}_{i=1}^m$ is uncountable. Conversely, There exists a super-self-conformal set A which generating IFS is uncountable. Moreover, the attractor for each IFS is different.

Proof. Let $(X, \{w_i\}_{i=1}^m)$ be a contractive conformal IFS with attractor E. And suppose X be a compact subset of R^n. $A = A(X)$ be defined as above. i.e. $A \supseteq \bigcup\limits_{i=1}^{m} w_i(A)$. To prove that A is the super-self-conformal sets for $(X, \{w_i\}_{i=1}^m)$, we need to assure A is compact only. By Lemma 3 it does. At the same time, there is uncountable super-self-conformal set generated by $(X, \{w_i\}_{i=1}^m)$, for X is arbitrary.

In addition, Lemma 4 gave the proof of the last part of the theorem.

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Tubular W-surfaces in 3-space

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Abstract In this paper, we studied the tubular W-surfaces that satisfy a Weingarten condition of type $ak_1 + bk_2 = c$, where a, b and c are constants and k_1 and k_2 denote the principal curvatures of M and M_i in Euclidean 3-space E^3 and Minkowski 3-space E_1^3, respectively.

Keywords Tubular surface, Weingarten surfaces, principal curvtures, Minkowski 3-space.

§1. Introduction and preliminaries

Classically, a Weingarten surface or linear Weingarten surface (or briefly, a W-surface) is a surface on which there is a nontrivial functional relation $\Phi(k_1, k_2) = 0$ between its principal curvatures k_1 and k_2 or equivalently, there is a nontrivial functional relation $\Phi(K, H) = 0$ between its Gaussian curvature K and mean curvature H. The existence of a nontrivial functional relation $\Phi(A, B) = 0$ such that Φ is of class C^1 is equivalent to the vanishing of the corresponding Jacobian determinant, namely, $\frac{\partial(A,B)}{\partial(s,t)} = 0$, where $(A, B) = (k_1, k_2)$ or (K, H) [4,5].

The set of solutions of this equation is also called the curvature diagram or the $W-$diagram of the surface. The study of Weingarten surfaces is a classical topic in differential geometry, as introduced by Weingarten in 1861. If the curvature diagram degenerates to exactly one point then the surface has two constant principal curvatures which is possible only for a piece of a plane, a sphere or a circular cylinder. If the curvature diagram is contained in one of the coordinate axes through the origin then the surface is developable. If the curvature diagram is contained in the main diagonal $k_1 = k_2$ then the surface is a piece of a plane or a sphere because every point is an umbilic. The curvature diagram is contained in a straight line parallel to the diagonal $k_1 = -k_2$ if and only if the mean curvature is constant. It is contained in a standard hyperbola $k_1 = \frac{c}{k_2}$ if and only the Gaussian curvature is constant [4].

D. W. Yoon and J. S. Ro studied tubes of (X, Y)-Weingarten type in Euclidean 3-space, where $X, Y \in \{K, H, K_{II}\}$ [1]. In this work we study Weingarten surfaces that satisfy the simplest case for Φ, that is, that Φ is of linear type: $ak_1 + bk_2 = c$, where a, b and c are constant with $a^2 + b^2 \neq 0$.

Following the Jacobi equation and the linear equation with respect to the principal curvatures k_1 and k_2, an interesting geometric question is raised: Classify all surfaces in Euclidean

3-space and Minkowski 3-space satisfying the conditions

$$\Phi(k_1, k_2) = 0, \tag{1.1}$$

$$ak_1 + bk_2 = c, \tag{1.2}$$

where $k_1 \neq k_2$ and $(a, b, c) \neq (0, 0, 0)$.

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle\ ,\rangle = -dx_1 + dx_3 + dx_3.$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since $\langle\ ,\rangle$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and null (lightlike) if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike) [2,4].

Minkowski space is originally from the relativity in Physics. In fact, a timelike curve corresponds to the path of an observer moving at less than the speed of light. Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^3.

We denote a surface M in E^3 and E_1^3 by

$$M(s,t) = (m_1(s,t), m_2(s,t), m_3(s,t)).$$

Let U be the standard unit normal vector field on a surface M defined by

$$U = \frac{M_s \wedge M_t}{\|M_s \wedge M_t\|}.$$

Then, the first fundamental form I and the second fundamental form II of a surface M are defined by

$$I = Eds^2 + 2Fdsdt + Gdt^2,$$

and

$$II = eds^2 + 2fdsdt + gdt^2,$$

respectively, where

$$E = \langle M_s, M_s\rangle, \quad F = \langle M_s, M_t\rangle, \quad G = \langle M_t, M_t\rangle.$$
$$e = \langle M_{ss}, U\rangle, \quad f = \langle M_{st}, U\rangle, \quad g = \langle M_{tt}, U\rangle.$$

[3,4]. On the other hand, the Gaussian curvature K and the mean curvature H are given by

$$K = \frac{eg - f^2}{EG - F^2}, \text{ and } H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}$$

respectively. The principal curvatures k_1 and k_2 are given by

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}.$$ [3]

In this paper, we would like to contribute the solution of the above question, by studying this question for tubes or tubular surfaces in Euclidean 3-space E^3 and Minkowski 3-space E_1^3.

§2. Tubular surfaces of Weingarten type in Euclidean 3-space

Definition 2.1. Let $\alpha : [a, b] \to E^3$ be a unit-speed curve. A tubular surface of radius $\lambda > 0$ about α is the surface with parametrization

$$M(s, \theta) = \alpha(s) + \lambda \left[N(s)\cos\theta + B(s)\sin\theta \right], a \leq s \leq b$$ [1].

We denote by κ and τ as the curvature and the torsion of the curve α. Then Frenet formulae of $\alpha(s)$ is defined by

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = -\tau N.$$

Furthermore, we have the natural frame $\{M_s, M_\theta\}$ given by

$$\begin{aligned} M_s &= (1 - \lambda\kappa\cos\theta)\, T - (\lambda\tau\sin\theta)\, N + (\lambda\tau\cos\theta)\, B, \\ M_\theta &= -(\lambda\sin\theta)\, N + (\lambda\cos\theta)\, B. \end{aligned}$$

From which the components of the first fundamental form are

$$E = \lambda^2\tau^2 + (1 - \lambda\kappa\cos\theta)^2, \quad F = \lambda^2\tau, \quad G = \lambda^2.$$

The unit normal vector field U is $U = -N\cos\theta - B\sin\theta$. The components of the second fundamental form of M are given by

$$e = -\lambda\tau^2 - (\kappa\cos\theta)(1 - \lambda\kappa\cos\theta), \quad f = \lambda\tau, \quad g = \lambda.$$

On the other hand, the Gauss curvature K, the mean curvature H are given by

$$K = -\frac{\kappa\cos\theta}{\lambda(1 - \lambda\kappa\cos\theta)}, \quad H = -\frac{(1 - 2\lambda\kappa\cos\theta)}{2\lambda(1 - \lambda\kappa\cos\theta)}$$

respectively. The principal curvatures are given by

$$k_1 = \frac{1}{\lambda}, \quad k_2 = -\frac{(1 - \lambda\kappa\cos\theta)}{2\lambda(1 - \lambda\kappa\cos\theta)}. \tag{2.1}$$

Differentiating k_1 and k_2 with respect to s and θ, we get

$$k_{1s} = 0, \quad k_{1\theta} = 0, \tag{2.2}$$

$$k_{2s} = -\frac{\kappa'\cos\theta}{(-1 + \lambda\kappa\cos\theta)^2}, \quad k_{2\theta} = \frac{\kappa\sin\theta}{(-1 + \lambda\kappa\cos\theta)^2}. \tag{2.3}$$

Now, we investigate a tubular surface M in E^3 satisfying the Jacobi equation $\Phi(k_1, k_2) = 0$. By using (2.2) and (2.3), M satisfies identically the Jacobi equation $k_{1s}k_{2\theta} - k_{1\theta}k_{2s} = 0$. Therefore, M is a Weingarten surface. We have the following theorem:

Theorem 2.2. A tubular surface M about unit-speed curve in E^3 is a Weingarten surface.

We suppose that M is a linear Weingarten surface in E^3. Thus, it satisfies the linear equation $ak_1 + bk_2 = c$. Then, by (2.1) and (2.2) we have

$$(c\lambda - b - a)\, \lambda\kappa\cos\theta + a - c\lambda = 0.$$

Since $\cos\theta$ and 1 are linearly independent, we get

$$(c\lambda - b - a)\,\lambda\kappa = 0, \quad a = c\lambda,$$

which imply

$$-b\lambda\kappa = 0.$$

If $b \neq 0$, then $\kappa = 0$. Thus, M is an open part of a circular cylinder in Euclidean 3-space. We have the following theorem and corollary:

Theorem 2.3. Let M be a tubular surface satisfying the linear equation $ak_1 + bk_2 = c$. If $b \neq 0$ and $\lambda \neq 0$, then it is an open part of a circular cylinder in Euclidean 3-space.

Corollary 2.4.

i. The surface M can not be minimal surface.

ii. The Mean curvature of the surface M is constant if and only if the curve $\alpha(s)$ is a straight line.

iii. The parallel surface of M is still a tubular surface.

iv. The surface M has not umbilic points.

§3. Tubular surfaces of Weingarten type in Minkowski 3-space

Definition 3.1. Let $\alpha : [a,b] \to E_1^3$ be a unit-speed spacelike curve with timelike principal normal. A tubular surface of radius $\lambda > 0$ about α is the surface with parametrization

$$M_1(s,\theta) = \alpha(s) + \lambda\left[N(s)\cosh\theta + B(s)\sinh\theta\right],$$

$a \leq s \leq b$, where $N(s), B(s)$ are timelike principal normal and spacelike binormal vectors to α, respectively [2].

Then Frenet formulae of $\alpha(s)$ is defined by

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = \tau N,$$

where $\langle T,T\rangle = \langle B,B\rangle = 1$, $\langle N,N\rangle = -1$, $\langle T,N\rangle = \langle N,B\rangle = \langle T,B\rangle = 0$ [2]. Furthermore, we have the natural frame $\{M_{1s}, M_{1\theta}\}$ given by

$$\begin{aligned} M_{1s} &= (1 + \lambda\kappa\cosh\theta)\,T + (\lambda\tau\sinh\theta)\,N + (\lambda\tau\cosh\theta)\,B, \\ M_{1\theta} &= (\lambda\sinh\theta)\,N + (\lambda\cosh\theta)\,B. \end{aligned}$$

From which the components of the first fundamental form are

$$E = \lambda^2\tau^2 + (1 + \lambda\kappa\cosh\theta)^2, \quad F = \lambda^2\tau, \quad G = \lambda^2.$$

The timelike unit normal vector field U_1 is $U_1 = N\cosh\theta + B\sinh\theta$. Since $\langle U_1, U_1\rangle = -1$, the surface M_1 is a spacelike surface. The components of the second fundamental form of M_1 are given by

$$e = -\left(\lambda\tau^2 + (\kappa\cosh\theta)(1 + \lambda\kappa\cosh\theta)\right), \quad f = -\lambda\tau, \quad g = -\lambda.$$

On the other hand, the Gauss curvature K, the mean curvature H are given by

$$K = \frac{\kappa \cosh\theta}{\lambda\left(1+\lambda\kappa\cosh\theta\right)}, \quad H = -\frac{\left(1+2\lambda\kappa\cosh\theta\right)}{2\lambda\left(1+\lambda\kappa\cosh\theta\right)},$$

respectively. The principal curvatures are given by

$$k_1 = -\frac{\kappa\cosh\theta}{1+\lambda\kappa\cosh\theta}, \quad k_2 = -\frac{1}{\lambda}. \tag{3.1}$$

Differentiating k_1 and k_2 with respect to s and θ, we get

$$k_{1s} = -\frac{\kappa'\cosh\theta}{\left(1+\lambda\kappa\cosh\theta\right)^2}, \quad k_{1\theta} = -\frac{\kappa\sinh\theta}{\left(1+\lambda\kappa\cosh\theta\right)^2}, \tag{3.2}$$

$$k_{2s} = 0, \quad k_{2\theta} = 0. \tag{3.3}$$

Now, we investigate a tubular surface M_1 in E_1^3 satisfying the Jacobi equation $\Phi(k_1, k_2) = 0$. By using (3.2) and (3.3), M_1 satisfies identically the Jacobi equation $k_{1s}k_{2\theta} - k_{1\theta}k_{2s} = 0$. Therefore, M_1 is a Weingarten surface. We have the following theorem:

Theorem 3.2. A tubular surface M_1 about unit-speed spacelike curve with timelike principal normal in Minkowski 3-space is a Weingarten surface.

We assume that a tubular surface M_1 is a linear Weingarten surface in E_1^3, such that, it satisfies the linear equation $ak_1 + bk_2 = c$. Then, by (3.1) we have

$$(b - a - c\lambda)\,\lambda\kappa\cosh\theta + b - c\lambda = 0.$$

Since $\cosh\theta$ and 1 are linearly independent, we get

$$(b - a - c\lambda)\,\lambda\kappa = 0, \quad b = c\lambda,$$

which imply

$$-a\lambda\kappa = 0.$$

If $a \neq 0$, then $\kappa = 0$. Thus, M_1 is an open part of a circular spacelike cylinder in Minkowski 3-space. We have the following theorem and corollary.

Theorem 3.3. Let M_1 be a tubular surface satisfying the linear equation $ak_1 + bk_2 = c$. If $a \neq 0$, then it is an open part of a circular spacelike cylinder in Minkowski 3-space.

Corollary 3.4.

i. The surface M_1 can not be minimal surface.

ii. The Mean curvature of the surface M_1 is constant if and only if the curve $\alpha(s)$ is a spacelike straight line.

iii. The parallel surface of M_1 is a spacelike tubular surface.

iv. The surface M_1 has not umbilic points.

Definition 3.5. Let $\alpha : [a, b] \to E_1^3$ be a unit-speed spacelike curve with spacelike principal normal. A tubular surface of radius $\lambda > 0$ about α is the surface with parametrization

$$M_2(s,\theta) = \alpha(s) + \lambda\left[N(s)\cosh\theta - B(s)\sinh\theta\right],$$

$a \leq s \leq b$, where $N(s)$, $B(s)$ are spacelike principal normal and timelike binormal vectors to α, respectively [2].

Frenet formulae of $\alpha(s)$ is defined by

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = \tau N,$$

where $\langle T, T\rangle = \langle N, N\rangle = 1$, $\langle B, B\rangle = -1$, $\langle T, N\rangle = \langle N, B\rangle = \langle T, B\rangle = 0$ [2]. Furthermore, we have the natural frame $\{M_{2s}, M_{2\theta}\}$ given by

$$\begin{aligned} M_{2s} &= (1 - \lambda\kappa \cosh\theta)\, T - (\lambda\tau \sinh\theta)\, N + (\lambda\tau \cosh\theta)\, B, \\ M_{2\theta} &= (\lambda \sinh\theta)\, N - (\lambda \cosh\theta)\, B. \end{aligned}$$

From which the components of the first fundamental form are

$$E = -\lambda^2\tau^2 + (1 - \lambda\kappa \cosh\theta)^2, \quad F = \lambda^2\tau, \quad G = -\lambda^2.$$

The spacelike unit normal vector field U_2 is $U_2 = N \cosh\theta - B \sinh\theta$. Since $\langle U_2, U_2\rangle = 1$, the surface M_2 is a timelike surface. Hence the components of the second fundamental form of M_2 are given by

$$e = \left(\lambda\tau^2 + (\kappa \cosh\theta)\,(1 - \lambda\kappa \cosh\theta)\right), \quad f = -\lambda\tau, \quad g = \lambda.$$

On the other hand, the Gauss curvature K, the mean curvature H are given by, respectively

$$K = -\frac{\kappa \cosh\theta}{\lambda\,(1 - \lambda\kappa \cosh\theta)}, \quad H = \frac{-1 + 2\lambda\kappa \cosh\theta}{2\lambda\,(1 - \lambda\kappa \cosh\theta)}.$$

The principal curvatures are given by

$$k_1 = -\frac{1}{\lambda}, \quad k_2 = \frac{\kappa \cosh\theta}{1 - \lambda\kappa \cosh\theta}. \tag{3.4}$$

Differentiating k_1 and k_2 respect to s and θ, we get

$$k_{1s} = 0, \quad k_{1\theta} = 0, \tag{3.5}$$

$$k_{2s} = \frac{\kappa' \cosh\theta}{(1 - \lambda\kappa \cosh\theta)^2}, \quad k_{2\theta} = \frac{\kappa \sinh\theta}{(1 - \lambda\kappa \cosh\theta)^2}. \tag{3.6}$$

Now, we investigate a tubular surface M_2 in E^3 satisfying the Jacobi equation $\Phi(k_1, k_2) = 0$. By using (3.5) and (3.6), M_2 satisfies identically the Jacobi equation $k_{1s}k_{2\theta} - k_{1\theta}k_{2s} = 0$. Therefore, M_2 is a Weingarten surface. We have the following theorem:

Theorem 3.6. A tubular surface M_2 about unit-speed spacelike curve with spacelike principal normal in Minkowski 3-space is a Weingarten surface.

We suppose that M_2 is a linear Weingarten surface in E_1^3. Then, it satisfies the linear equation $ak_1 + bk_2 = c$. Thus, by using (3.4), we obtained

$$(a + b + c\lambda)\, \lambda\kappa \cosh\theta - a - c\lambda = 0.$$

Since $\cosh\theta$ and 1 are linearly independent, we get

$$(a + b + c\lambda)\, \lambda\kappa = 0, \quad a = -c\lambda,$$

which imply

$$b\lambda\kappa = 0.$$

If $b \neq 0$, then $\kappa = 0$. Thus, M_2 is an open part of a circular timelike cylinder in Minkowski 3-space. We have the following theorem and corollary.

Theorem 3.7. Let M_2 be a tubular surface satisfying the linear equation $ak_1 + bk_2 = c$. If $b \neq 0$ and $\lambda \neq 0$, then it is an open part of a circular timelike cylinder in Minkowski 3-space.

Corollary 3.8.

i. The surface M_2 can not be minimal surface.

ii. The mean curvature of the surface M_2 is constant if and only if the curve $\alpha(s)$ is a spacelike straight line.

iii. The parallel surface of M_2 is a timelike tubular surface.

iv. The surface M_2 has not umbilic points.

Definition 3.9. Let $\alpha : [a, b] \to E_1^3$ be a unit-speed timelike curve. A tubular surface of radius $\lambda > 0$ about α is the surface with parametrization

$$M_3(s, \theta) = \alpha(s) + \lambda \left[N(s) \cos\theta + B(s) \sin\theta\right],$$

$a \leq s \leq b$, where $N(s)$, $B(s)$ are spacelike principal normal and spacelike binormal vectors to α, respectively [2].

Then Frenet formulae of $\alpha(s)$ is defined by

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = -\tau N,$$

where $\langle T, T\rangle = -1$, $\langle N, N\rangle = \langle B, B\rangle = 1$, $\langle T, N\rangle = \langle N, B\rangle = \langle T, B\rangle = 0$. Furthermore, we have the natural frame $\{M_{3s}, M_{3\theta}\}$ given by

$$\begin{aligned} M_{3s} &= (1 + \lambda\kappa \cos\theta)\, T - (\lambda\tau \sin\theta)\, N + (\lambda\tau \cos\theta)\, B, \\ M_{3\theta} &= -(\lambda \sin\theta)\, N + (\lambda \cos\theta)\, B. \end{aligned}$$

From which the components of the first fundamental form are

$$E = \lambda^2\tau^2 - (1 + \lambda\kappa \cos\theta)^2, \ F = \lambda^2\tau, \ G = \lambda^2.$$

The spacelike unit normal vector field U_3 is $U_3 = N \cos\theta + B \sin\theta$. Since $\langle U_3, U_3\rangle = 1$, the surface M_3 is a timelike surface. Thus the components of the second fundamental form of M_3 are given by

$$e = \left(-\lambda\tau^2 + (\kappa \cos\theta)(1 + \lambda\kappa \cos\theta)\right), \ f = -\lambda\tau, \ g = -\lambda.$$

On the other hand, the Gauss curvature K, the mean curvature H are given by, respectively

$$K = \frac{\kappa \cos\theta}{\lambda (1 + \lambda\kappa \cos\theta)}, \quad H = -\frac{1 + 2\lambda\kappa \cos\theta}{2\lambda (1 + \lambda\kappa \cos\theta)}.$$

The principal curvatures are given by

$$k_1 = -\frac{1}{\lambda}, \quad k_2 = -\frac{\kappa \cos\theta}{1 + \lambda\kappa \cos\theta}. \tag{3.7}$$

Differentiating k_1 and k_2 with respect to s and θ, we get

$$k_{1s} = 0, \quad k_{1\theta} = 0, \tag{3.8}$$

$$k_{2s} = -\frac{\kappa' \cos\theta}{(1 + \lambda\kappa\cos\theta)^2}, \quad k_{2\theta} = \frac{\kappa \sin\theta}{(1 + \lambda\kappa\cos\theta)^2}. \tag{3.9}$$

Now, we investigate a tubular surface M in E^3 satisfying the Jacobi equation $\Phi(k_1, k_2) = 0$. By using (3.8) and (3.9), M_3 satisfies identically the Jacobi equation $k_{1s}k_{2\theta} - k_{1\theta}k_{2s} = 0$. Therefore, M_3 is a Weingarten surface. We have the following theorem:

Theorem 3.10. A tubular surface M_3 about unit-speed timelike curve in Minkowski 3-space is a Weingarten surface.

Assume that a tubular surface M_3 in E_1^3 is a linear Weingarten surface. Then it satisfies the linear equation $ak_1 + bk_2 = c$. Thus, by using (3.7), we obtained

$$(a + b + c\lambda)\,\lambda\kappa\cos\theta + a + c\lambda = 0.$$

Since $\cos\theta$ and 1 are linearly independent, we get

$$(a + b + c\lambda)\,\lambda\kappa = 0, \quad a = -c\lambda,$$

which imply

$$b\lambda\kappa = 0.$$

If $b \neq 0$, then $\kappa = 0$. Thus, M_3 is an open part of a circular timelike cylinder in Minkowski 3-space. We have the following theorem and corollary.

Theorem 3.11. Let M_3 be a tubular surface satisfying the linear equation $ak_1 + bk_2 = c$. If $b \neq 0$, then it is an open part of a circular timelike cylinder in Minkowski 3-space.

Corollary 3.12.

i. The surface M_3 can not be minimal surface.

ii. The mean curvature of the surface M_3 is constant if and only if the curve $\alpha(s)$ is a timelike straight line.

iii. The parallel surface of M_3 is a timelike tubular surface.

iv. The surface M_3 has not umbilic points.

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Integral formula in special three-dimansional Kenmotsu manifold $\mathbb{K}$ with $\eta-$ parallel ricci tensor

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Abstract In this paper, we study minimal surfaces for simply connected immersed minimal surfaces in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor. We consider the Riemannian left invariant metric and use some results of Levi-Civita connection.
Keywords Weierstrass representation, Kenmotsu manifold, minimal surface.

§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [4]; the non-standard will be given in this paper as and when The study of minimal surfaces played a formative role in the development of mathematics over the last two centuries. Today, minimal surfaces appear in various guises in diverse areas of mathematics, physics, chemistry and computer graphics, but have also been used in differential geometry to study basic properties of immersed surfaces in contact manifolds.

Minimal surface, such as soap film, has zero curvature at every point. It has attracted the attention for both mathematicians and natural scientists for different reasons. Mathematicians are interested in studying minimal surfaces that have certain properties, such as completedness and finite total curvature, while scientists are more inclined to periodic minimal surfaces observed in crystals or biosystems such as lipid bilayers.

Weierstrass representations are very useful and suitable tools for the systematic study of minimal surfaces immersed in n-dimensional spaces. This subject has a long and rich history. It has been extensively investigated since the initial works of Weierstrass [19]. In the literature there exists a great number of applications of the Weierstrass representation to various domains of Mathematics, Physics, Chemistry and Biology. In particular in such areas as quantum field theory [8], statistical physics [14], chemical physics, fluid dynamics and membranes [16], minimal surfaces play an essential role. More recently it is worth mentioning that works by Kenmotsu [10], Hoffmann [9], Osserman [15], Budinich [5], Konopelchenko [6,11] and Bobenko [3,4] have made very significant contributions to constructing minimal surfaces in a systematic way and to understanding their intrinsic geometric properties as well as their integrable dynamics. The type of extension of the Weierstrass representation which has been useful in three-dimensional applications to multidimensional spaces will continue to generate many additional applications

to physics and mathematics.

In this paper, we study minimal surfaces for simply connected immersed minimal surfaces in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor. We consider the Riemannian left invariant metric and use some results of Levi-Civita connection.

§2. Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold with 1-form η, the associated vector field ξ, (1.1)-tensor field ϕ and the associated Riemannian metric g. It is well known that [2]

$$\phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X, Y on M. Moreover,

$$(\nabla_X \phi) Y = -\eta(Y)\phi(X) - g(X, \phi Y)\xi, \qquad X, Y \in \chi(M), \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g, then (M, ϕ, ξ, η, g) is called an almost Kenmotsu manifold [2].

In Kenmotsu manifolds the following relations hold [2]:

$$\nabla_X \eta Y = g(\phi X, \phi Y), \tag{2.7}$$

§3. Special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor

Definition 3.1. The Ricci tensor S of a Kenmotsu manifold is called η-parallel if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0.$$

We consider the three-dimensional manifold

$$\mathbb{K} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2, x_3) \neq (0, 0, 0) \right\},$$

where (x_1, x_2, x_3) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$\mathbf{e}_1 = x_3 \frac{\partial}{\partial x_1}, \quad \mathbf{e}_2 = x_3 \frac{\partial}{\partial x_2}, \quad \mathbf{e}_3 = -x_3 \frac{\partial}{\partial x_3} \tag{3.1}$$

are linearly independent at each point of $\mathbb{K}$. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1 \tag{3.2}$$

$$g(e_1,e_2)=g(e_2,e_3)=g(e_3,e_1)=1$$

The characterising properties of $\chi(\mathbb{K})$ are the following commutation relations:

$$[\mathbf{e}_1,\mathbf{e}_2]=0,\quad [\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_1,\quad [\mathbf{e}_2,\mathbf{e}_3]=\mathbf{e}_2. \tag{3.3}$$

Let η be the 1-form defined by

$$\eta(Z)=g(Z,\mathbf{e}_3),\ \text{for any } Z\in\chi(M).$$

Let be the (1.1) tensor field defined by

$$\phi(\mathbf{e}_1)=-\mathbf{e}_2,\ \phi(\mathbf{e}_2)=\mathbf{e}_1,\ \phi(\mathbf{e}_3)=0.$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3)=1, \tag{3.4}$$

$$\phi^2(Z)=-Z+\ \eta(Z)\mathbf{e}_3, \tag{3.5}$$

$$g\left(\phi Z,\phi W\right)=g\left(Z,W\right)-\eta(Z)\eta(W), \tag{3.6}$$

for any $Z,W\in\chi(M)$. Thus for $\mathbf{e}_3=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on $\mathbb{M}$.

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned}2g\left(\nabla_X Y,Z\right) &= Xg\left(Y,Z\right)+Yg\left(Z,X\right)-Zg\left(X,Y\right)\\ &\quad -g\left(X,[Y,Z]\right)-g\left(Y,[X,Z]\right)+g\left(Z,[X,Y]\right),\end{aligned}$$

which is known as Koszul's formula.

Proposition 3.2. For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, *defined above the following is true:*

$$\nabla=\begin{pmatrix}0&0&\mathbf{e}_1\\0&0&\mathbf{e}_2\\0&0&0\end{pmatrix}, \tag{3.7}$$

where the (i,j) -element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k,k=1,2,3\}=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

Then, we write the Kozul formula for the Levi–Civita connection is:

$$2g(\nabla_{\mathbf{e}_i}\mathbf{e}_j,\mathbf{e}_k)=L^k_{ij}.$$

From (3.7), we get

$$L^1_{13}=2,\ L^2_{23}=2. \tag{3.8}$$

§4. Weierstrass representation formula in special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel ricci tensor

$\Sigma \subset \mathbb{K}$ be a surface and $\wp : \Sigma \longrightarrow \mathbb{K}$ a smooth map. The pull-back bundle $\wp^*(T\mathbb{K})$ has a metric and compatible connection, the pull-back connection, induced by the Riemannian metric and the Levi–Civita connection of $\mathbb{K}$. Consider the complexified bundle $\mathbb{E} = \wp^*(T\mathbb{K}) \otimes \mathbb{C}$.

Let (u, v) be local coordinates on Σ, and $z = u + iv$ the (local) complex parameter and set, as usual,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{4.1}$$

Let

$$\frac{\partial \wp}{\partial u} |_p = \wp_{*p}\left(\frac{\partial}{\partial u} |_p\right), \quad \frac{\partial \wp}{\partial v} |_p = \wp_{*p}\left(\frac{\partial}{\partial v} |_p\right), \tag{4.2}$$

and

$$\phi = \wp_z = \frac{\partial \wp}{\partial z} = \frac{1}{2}\left(\frac{\partial \wp}{\partial u} - i\frac{\partial \wp}{\partial v}\right). \tag{4.3}$$

Let now $\wp : \Sigma \longrightarrow \mathbb{K}$ be a conformal immersion and $z = u + iv$ a local conformal parameter. Then, the induced metric is

$$ds^2 = \lambda^2(du^2 - dv^2) = \lambda^2 |dz|^2, \tag{4.4}$$

and the Beltrami–Laplace operator on $\mathbb{K}$, with respect to the induced metric, is given by

$$\Delta = \lambda^{-2}\left(\frac{\partial}{\partial u}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\frac{\partial}{\partial v}\right). \tag{4.5}$$

We recall that a map $\wp : \Sigma \longrightarrow \mathbb{K}$ is harmonic if its tension field

$$\tau(\wp) = \texttt{trace}\nabla d\wp = 0. \tag{4.6}$$

Let $\{x_1, x_2, x_3\}$ be a system of local coordinates in a neighborhood U of M such that $U \cap \wp(\Sigma) \neq \emptyset$. Then, in an open set $G \subset \Sigma$

$$\phi = \sum_{j=1}^{3} \phi_j \frac{\partial}{\partial x_j}, \tag{4.7}$$

for some complex-valued functions ϕ_j defined on G. With respect to the local decomposition of ϕ, the tension field can be written as

$$\tau(\zeta) = \sum_i \left\{ \Delta \wp_i + 4\lambda^{-2} \sum_{j,k=1}^{n} \Gamma^i_{jk} \frac{\partial \wp_j}{\partial \bar{z}} \frac{\partial \wp_k}{\partial z} \right\} \frac{\partial}{\partial x_i}, \tag{4.8}$$

where Γ^i_{jk} are the Christoffel symbols of $\mathbb{K}$.

From (4.3), we have

$$\tau(\wp) = 4\lambda^{-2} \sum_i \left\{ \frac{\partial \phi_i}{\partial \bar{z}} + \sum_{j,k=1}^{n} \Gamma^i_{jk} \bar{\phi}_j \phi_k \right\} \frac{\partial}{\partial x_i}.$$

The section ϕ is holomorphic if and only if

$$\nabla_{\frac{\partial}{\partial \bar{z}}}\left(\sum_{i=1}^{3}\phi_i\frac{\partial}{\partial x_i}\right)=\sum_i\left\{\frac{\partial\phi_i}{\partial\bar{z}}\frac{\partial}{\partial x_i}+\phi_i\nabla_{\frac{\partial\varsigma}{\partial\bar{z}}}\frac{\partial}{\partial x_i}\right\}.$$

Using (4.3), we get

$$\nabla_{\frac{\partial}{\partial \bar{z}}}\left(\sum_{i=1}^{3}\phi_i\frac{\partial}{\partial x_i}\right)=\sum_i\left\{\frac{\partial\phi_i}{\partial\bar{z}}\frac{\partial}{\partial x_i}+\phi_i\nabla_{\sum_j\bar{\phi}_j\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i}\right\}.$$

Making necessary calculations, we obtain

$$\nabla_{\frac{\partial}{\partial \bar{z}}}\left(\sum_{i=1}^{3}\phi_i\frac{\partial}{\partial x_i}\right)=\sum_i\left\{\frac{\partial\phi_i}{\partial\bar{z}}+\sum_{j,k}\Gamma^i_{jk}\overline{\phi}_j\phi_k\right\}\frac{\partial}{\partial x_i}=0.$$

Thus, ϕ is holomorphic if and only if

$$\frac{\partial\phi_i}{\partial\bar{z}}+\sum_{j,k}\Gamma^i_{jk}\overline{\phi}_j\phi_k=0,\quad i=1,2,3. \tag{4.9}$$

Theorem 4.1. (Weierstrass representation) Let $\mathbb{K}$ be the group of rigid motions of Euclidean 2-space and $\{x_1,x_2,x_3\}$ local coordinates. Let $\phi_j,\ j=1,2,3$ be complex-valued functions in an open simply connected domain $G\subset\mathbb{C}$ which are solutions of (4.9). Then, the map

$$\wp_j(u,v)=2Re\left(\int_{z_0}^{z}\phi_j dz\right) \tag{4.10}$$

is well defined and defines a maximal conformal immersion if and only if the following conditions are satisfied :

$$\sum_{j,k=1}^{3}g_{ij}\phi_j\overline{\phi_k}\neq 0 \text{ and } \sum_{j,k=1}^{3}g_{ij}\phi_j\phi_k=0\ .$$

Let us expand Υ with respect to this basis to obtain

$$\Upsilon=\sum_{k=1}^{3}\psi_k\mathbf{e}_k\ . \tag{4.11}$$

Setting

$$\phi=\sum_i\phi_i\frac{\partial}{\partial x_i}=\sum_i\psi_i e_i, \tag{4.12}$$

for some complex functions $\phi_i,\psi_i:G\subset\mathbb{C}$. Moreover, there exists an invertible matrix $A=(A_{ij})$, with function entries $A_{ij}:\wp(G)\cap U\to\mathbb{R}$, $i,j=1,2,3$, such that

$$\phi_i=\sum_j A_{ij}\psi_j. \tag{4.13}$$

Using the expression of ϕ, the section ϕ is holomorphic if and only if

$$\frac{\partial\psi_i}{\partial\bar{z}}+\frac{1}{2}\sum_{j,k}L^i_{jk}\overline{\psi}_j\psi_k=0,\quad i=1,2,3. \tag{4.14}$$

Theorem 4.2. Let ψ_j, $j = 1, 2, 3$, be complex-valued functions defined in a open simply connected set $G \subset \mathbb{C}$, such that the following conditions are satisfied :

i. $|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 \neq 0$,
ii. $\psi_1^2 + \psi_2^2 + \psi_3^2 = 0$,
iii. ψ_j *are solutions of (4.14).*
Then, the map $\wp : G \to \mathbb{K}$ defined by

$$\wp_i(u, v) = 2Re\left(\int_{z_0}^{z} \sum_j A_{ij}\psi_j dz\right) \tag{4.15}$$

is a conformal minimal immersion.

Proof. By Theorem 4.1 we see that $\wp$ is a harmonic map if and only if $\wp$ satisfy (4.15). Then, the map $\wp$ is a conformal maximal immersion.

Since the parameter z is conformal, we have

$$\langle \Upsilon, \Upsilon \rangle = 0, \tag{4.16}$$

which is rewritten as

$$\psi_1^2 + \psi_2^2 + \psi_3^2 = 0. \tag{4.17}$$

From (4.17), we have

$$(\psi_1 - i\psi_2)(\psi_1 + i\psi_2) = -\psi_3^2, \tag{4.18}$$

which suggests the definition of two new complex functions

$$\Omega := \sqrt{\frac{1}{2}(\psi_1 - i\psi_2)}, \quad \Phi := \sqrt{-\frac{1}{2}(\psi_1 + i\psi_2)}. \tag{4.19}$$

The functions Ω and Φ are single-valued complex functions which, for suitably chosen square roots, satisfy

$$\begin{aligned} \psi_1 &= \Omega^2 - \Phi^2, \\ \psi_2 &= i\left(\Omega^2 + \Phi^2\right), \\ \psi_3 &= 2\Omega\Phi. \end{aligned} \tag{4.20}$$

Lemma 4.3. *If* Υ *satisfies the equation (4.14), then*

$$\Omega\Omega_{\bar{z}} - \Phi\Phi_{\bar{z}} = -\left(|\Omega|\,\Phi\bar{\Omega} - |\Phi|\,\Omega\bar{\Phi}\right), \tag{4.21}$$

$$\Omega\Omega_{\bar{z}} + \Phi\Phi_{\bar{z}} = \left(|\Omega|\,\Phi\bar{\Omega} + |\Phi|\,\Omega\bar{\Phi}\right), \tag{4.22}$$

$$\Omega_{\bar{z}}\Phi + \Omega\Phi_{\bar{z}} = 0. \tag{4.23}$$

Proof. Using (4.14) and (4.20), we have

$$\frac{\partial\psi_1}{\partial\bar{z}} = -\overline{\psi}_1\psi_3,$$

$$\frac{\partial\psi_2}{\partial\bar{z}} = -\overline{\psi}_2\psi_3,$$

$$\frac{\partial\psi_3}{\partial\bar{z}} = 0.$$

Substituting (4.20) into (4.24), we have (4.21)-(4.23).

Corollary 4.4.

$$\Omega_{\bar{z}} = |\Phi|\,\bar{\Phi} \tag{4.25}$$

Corollary 4.5.

$$\Phi_{\bar{z}} = |\Omega|\,\bar{\Omega}. \tag{4.26}$$

Theorem 4.6. Let Ω and Φ be complex-valued functions defined in a simply connected domain $G \subset \mathbb{C}$. Then the map $\wp : G \to \mathbb{K}$, defined by

$$\begin{aligned}\wp_1(u,v) &= Re\left(\int_{z_0}^{z} x_3\left[\Omega^2+\Phi^2\right]dz\right),\\ \wp_2(u,v) &= Re\left(\int_{z_0}^{z} ix_3\left(\Omega^2-\Phi^2\right)dz\right),\\ \wp_3(u,v) &= -Re\left(\int_{z_0}^{z}(2x_3\Omega\Phi)dz\right),\end{aligned} \tag{4.27}$$

is a conformal minimal immersion.

Proof. Using (4.12), we get

$$\phi_1 = x_3\psi_1,\ \phi_2 = x_3\psi_2,\ \phi_3 = -x_3\psi_3. \tag{4.28}$$

From (4.10) we have the system (4.27). Using Theorem 4.2 $\wp : G \to \mathbb{K}$ is a conformal minimal immersion.

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Iterative method for the solution of linear matrix equation $AXB^T + CYD^T = F$ [1]

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Abstract In this paper, we obtain indirectly the solution of matrix equation $AXB^T + CYD^T = F$, by establishing iterative method of the constrained matrix equation $MZN^T = F$ with $PZP = Z$, where $M = (A, C)U$, $N = (B, D)U$, and the matrix P related to orthogonal matrix U is symmetric orthogonal. Furthermore, when matrix equation $AXB^T + CYD^T = F$ is consistent, for given matrices X_0, Y_0, the nearness matrix pair $(\hat{X}, \hat{Y})$ of matrix equation $AXB^T + CYD^T = F$ can be obtained. Finally, given numerical examples and associated convergent curves show that this iterative method is efficient.

Keywords Iterative method, matrix equation, least-norm solution, generalized centro-symm -etric solution, optimal approximation.

§1. Introduction

Some notations will be used. Let $R^{m\times n}$ be the set of all $m \times n$ real matrices, $OR^{n\times n}$ and $SOR^{n\times n}$ be the sets of all orthogonal matrices and symmetric orthogonal matrices in $R^{n\times n}$, respectively. For a matrix $A = (a_1, a_2, \dots, a_n) \in R^{m\times n}$, $a_i \in R^m$, $i = 1, 2, \dots, n$. Let A^T, $tr(A)$ and $R(A)$ be the transpose, trace and the column space of A, respectively. The symbol $vec(\cdot)$ denotes the vec operator, i.e., $vec(A) = (a_1^T, a_2^T, \dots, a_n^T)^T$. Define $< A, B >= tr(B^T A)$ as the inner product of matrices A and B with appropriate sizes, which generates the Frobenius norm, i.e. $\|A\| = \sqrt{< A, A >} = \sqrt{tr(A^T A)}$. $A \otimes B$ stands for the Kronecker product of matrices A and B.

In this paper, we consider the following problems.

Problem I. Given $A \in R^{m\times p}$, $B \in R^{n\times p}$, $C \in R^{m\times q}$, $D \in R^{n\times q}$, $F \in R^{m\times n}$, find $X \in R^{p\times p}$, $Y \in R^{q\times q}$ such that

$$AXB^T + CYD^T = F. \tag{1}$$

Let S_E be the solution set of matrix equation (1).

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Problem II. When S_E is not empty, for given $X_0 \in R^{p\times p}$, $Y_0 \in R^{q\times q}$, find $(\hat{X}, \hat{Y}) \in S_E$ such that

$$\|(\hat{X}, \hat{Y}) - (X_0, Y_0)\| = \min_{(X,Y)\in S_E} \|(X, Y) - (X_0, Y_0)\|.$$

There have been many elegant results on the matrix equation $AXB^T + CYD^T = F$ or its particular forms, with unknown matrices X and Y. Such as, the matrix equation $AX - YB = C$ has been investigated by Baksalary and Kala [1], Flanders and Wimmer [2] have given the necessary and sufficient conditions for its consistency and general solution, respectively. For the matrix equation (1), the solvability conditions and general solution have been derived in [3, 5, 6] by means of g-inverse, singular value decomposition (SVD), generalized SVD (GSVD) [4] or the canonical correlation decomposition (CCD) of matrix pairs, respectively. Especially, the symmetric solution of matrix equation $AXA^T + BYB^T = C$ has been represented in [7] by GSVD. In addition, as the particular form of matrix equation (1), Peng et al. [8, 9] have respectively obtained the symmetric solution and the least-squares symmetric solution of $AXB = C$ by iterative methods based on the classical Conjugate Gradient Method; Y. X. Peng et al. [10] have considered the symmetric solution of a pair of simultaneous linear matrix equation $A_1XB_1 = C_1, A_2XB_2 = C_2$. Moreover, the generation solution of matrix equation $AXB + CX^TD = E$ and $AXB + CYD = F$ have also been studied by establishing iterative method in [11, 12]. For any initial iterative matrix, they have showed that the associated solutions can be obtained within finite iteration steps in the absence of roundoff errors.

The optimal approximation Problem II occurs frequently in experimental design, see for instance [13, 14]. Here the matrix pair (X_0, Y_0), may be obtained from experiments, and does not satisfy usually the needed forms and minimum residual requirements. Whereas the matrix pair $(\hat{X}, \hat{Y})$ is the ones that satisfies the needed forms and the minimum residual restrictions.

Motivated by the iterative methods presented in [8-11], in this paper, we reconsider the above two problems by another way that is different from that mentioned in [12]. By choosing an arbitrary unitary matrix U, Problem I is equivalently transformed into solving the matrix equation $MZN^T = F$ with $PZP = Z$ constraint, where $M = (A,\, C)U$, $N = (B,\, D)U$, and P related to U is a given symmetric and orthogonal matrix, then Problem I and Problem II can be transformed equivalently into Problems A and B (which will be stated rearwards), respectively. For the Problems A and B, we will find their solutions by establishing iterative algorithm, then, the solutions of Problems I and II can be obtained naturally.

We need the following conception (see [15] for details).

Definition 1.1. Given matrix $P \in SOR^{n\times n}$, we say that a matrix $A \in R^{n\times n}$ is generalized centro-symmetric (or generalized central anti-symmetric) with respect to P, if $PAP = A$ (or $PAP = -A$). The set of $n \times n$ generalized centro-symmetric (or generalized central anti-symmetric) matrices with respect to P is denoted by $CSR_P^{n\times n}$ (or $CASR_P^{n\times n}$).

Obviously, if $P = J$, $J = (e_n, e_{n-1}, \ldots, e_2, e_1)$, where e_i is the i^{th} column of identity matrix I_n, the generalized centro-symmetric matrices become centro-symmetric matrices which are widely applied in information theory, numerical analysis, linear estimate theory and so on (see, e.g., [16, 17, 18]).

Definition 1.2. Let matrices $M, N \in R^{s\times t}$, where s, t are arbitrary integers. We say that matrices M, N are orthogonal each other, if $tr(M^TN) = 0$.

The symmetric orthogonal matrices have been discussed in many literatures, and they possesses special structural peculiarity, which can be represented as follows.

Lemma 1.1.[15] If $P \in SOR^{l\times l}$, and $p = \mathrm{rank}(\frac{1}{2}(I_l + P))$, then P can be expressed as

$$P = U^T \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} U, \tag{2}$$

where $q = l - p$, $U \in OR^{l\times l}$.

Employing Lemma 1.1 and Definition 1.1, we get the following assertion.

Lemma 1.2. Given symmetric orthogonal matrix P with the form of (2). Then $A \in CSR_P^{l\times l}$ if and only if

$$A = U^T \begin{pmatrix} X & \\ & Y \end{pmatrix} U,$$

where $X \in R^{p\times p}$, $Y \in R^{q\times q}$ are arbitrary.

For arbitrary orthogonal matrix $U \in R^{l\times l}$, let symmetric orthogonal matrix P satisfy $p = \mathrm{rank}(\frac{I_l+P}{2})$ (the order of unknown matrix X in (1)) as in Lemma 1.1. Denote

$$M = (A\ C)U \in R^{m\times l},\ N = (B\ D)U \in R^{n\times l},$$

then

$$Z = U^T \begin{pmatrix} X & \\ & Y \end{pmatrix} U \in CSR_P^{l\times l},$$

where A, B, C, D are the matrices mentioned in Problem I, and $X \in R^{p\times p}$, $Y \in R^{q\times q}$. It is easy to verify that the solvability of matrix equation (1) is equivalent to that of the constrained matrix equation

$$MZN^T = F, Z \in CSR_P^{l\times l}. \tag{3}$$

Therefore, for above matrices M, N, F, Problem 1 and Problem II can be transformed equivalently into the following Problems A and B, respectively.

Problem A. Given $M \in R^{m\times l}$, $N \in R^{n\times l}$, $F \in R^{m\times n}$, and $P \in SOR^{l\times l}$, find $Z \in CSR_P^{l\times l}$ such that (3) holds.

Problem B. Let matrix $Z_0 = U^T \begin{pmatrix} X_0 & \\ & Y_0 \end{pmatrix} U \in CSR_P^{l\times l}$, where X_0, Y_0 as in Problem II, find $\hat{Z} \in S$, such that

$$\|\hat{Z} - Z_0\| = \min_{Z\in S} \|Z - Z_0\|,$$

where S stands for the solution set of Problem A.

This paper is organized as follows. In section 2, an iterative algorithm will be proposed to solve Problem A, and the solution to Problem I can be obtained naturally. In section 3, we will find the solution to Problem II by using this algorithm. In section 4, some numerical examples will be given to validate our results.

§2. The iterative method for solving Problem I

The iterative algorithm for Problem I (A) can be expressed as follows.

Algorithm 2.1.

Step 1: Input matrices $A \in R^{m\times p}$, $B \in R^{n\times p}$, $C \in R^{m\times q}$, $D \in R^{n\times q}$, $F \in R^{m\times n}$, and partition $P \in SOR^{l\times l}$ as in (2). Let $M = (A \quad C)U \in R^{m\times l}$, $N = (B \quad D)U \in R^{n\times l}$. For arbitrary matrices $X_1 \in R^{p\times p}$, $Y_1 \in R^{q\times q}$, denote $Z_1 = U^T \begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix} U$.

Step 2: Calculate

$R_1 = F - MZ_1N^T$,

$P_1 = \frac{1}{2}(M^TR_1N + PM^TR_1NP)$,

$k := 1$.

Step 3: Calculate

$Z_{k+1} = Z_k + \frac{\| R_k \|^2}{\| P_k \|^2}P_k$.

Step 4: Calculate

$$\begin{aligned} R_{k+1} &= F - MZ_{k+1}N^T \\ &= R_k - \frac{\| R_k \|^2}{\| P_k \|^2}MP_kN^T, \end{aligned}$$

$P_{k+1} = \frac{1}{2}(M^TR_{k+1}N + PM^TR_{k+1}NP) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}P_k$.

If $R_{k+1} = 0$, or $R_{k+1} \neq 0$, $P_{k+1} = 0$, stop. Otherwise, let $k := k + 1$, go to Step 3.

Remark 1. (a) It is clear that X_i, $P_i \in CSR_P^{n\times n}$ in Algorithm 2.1.

(b) If $R_k = 0$ in the iteration process, then Z_k is a solution of Problem A. Based on the analysis in section I, a solution pair (X_k, Y_k) of Problem I can be derived from

$$UZ_kU^T = \begin{pmatrix} X_k & \\ & Y_k \end{pmatrix}.$$

In the sequel, we analysis the convergency of Algorithm 2.1.

Lemma 2.1. The sequences $\{R_i\}$, $\{P_i\}$ $(i = 1, 2, \ldots)$, generated by Algorithm 2.1, satisfy that

$$tr(R_{i+1}^TR_j) = tr(R_i^TR_j) - \frac{\|R_i\|^2}{\|P_i\|^2}tr(P_i^TP_j) + \frac{\|R_i\|^2\|R_j\|^2}{\|P_i\|^2\|R_{j-1}\|^2}tr(P_i^TP_{j-1}). \quad (4)$$

Proof. From Lemmas 1.1, Algorithm 2.1, we obtain

$$\begin{aligned} tr[(MP_iN^T)^TR_j] &= tr(P_i^TM^TR_jN) \\ &= tr\left[P_i^T\left(\frac{M^TR_jN + PM^TR_jNP}{2} + \frac{M^TR_jN - PM^TR_jNP}{2}\right)\right] \\ &= tr\left(P_i^T\frac{M^TR_jN + PM^TR_jNP}{2}\right) \\ &= tr\left[P_i^T\left(P_j - \frac{\|R_j\|^2}{\|R_{j-1}\|^2}P_{j-1}\right)\right] \\ &= tr(P_i^TP_j) - \frac{\|R_j\|^2}{\|R_{j-1}\|^2}tr(P_i^TP_{j-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} tr(R_{i+1}^T R_j) &= tr\left[\left(R_i - \frac{\|R_i\|^2}{\|P_i\|^2} M P_i N^T\right)^T R_j\right] \\ &= tr(R_i, R_j) - \frac{\|R_i\|^2}{\|P_i\|^2} tr[(M P_i N^T)^T R_j] \\ &= tr(R_i, R_j) - \frac{\|R_i\|^2}{\|P_i\|^2} tr(P_i^T P_j) + \frac{\|R_i\|^2 \|R_j\|^2}{\|P_i\|^2 \|R_{j-1}\|^2} tr(P_i^T P_{j-1}). \end{aligned}$$

Thus the proof is completed.

The following conclusion is essential for our main results.

Lemma 2.2. The sequences $\{R_i\}$, $\{P_i\}$ generated by the iterative method satisfy that

$$tr(R_i^T R_j) = 0, tr(P_i^T P_j) = 0, \ i, j = 1, 2, \ldots, k \, (k \geq 2), \, i \neq j. \tag{5}$$

Proof. We prove this conclusion by induction.

When $k = 2$, resorting to Lemma 1.1, Algorithm 2.1 and the proof of Lemma 2.1 yields

$$\begin{aligned} tr(R_2^T R_1) &= tr(R_1^T R_1) - \frac{\| R_1 \|^2}{\| P_1 \|^2} tr[(M P_1 N^T)^T R_1) \\ &= tr(R_1^T R_1) - \frac{\| R_1 \|^2}{\| P_1 \|^2} tr(P_1^T P_1) = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} tr(P_2^T P_1) &= tr\left[\left(\frac{M^T R_2 N + P M^T R_2 N P}{2} + \frac{\|R_2\|^2}{\|R_1\|^2} P_1\right)^T P_1\right] \\ &= tr(R_2^T M P_1 N^T) + \frac{\|R_2\|^2}{\|R_1\|^2} tr(P_1^T P_1) \\ &= tr\left[R_2^T \frac{\|P_1\|^2}{\|R_1\|^2}(R_1 - R_2)\right] + \frac{\|R_2\|^2}{\|R_1\|^2} tr(P_1^T P_1) g \\ &= 0. \end{aligned} \tag{7}$$

Assume that (5) holds for $k = s$, that is, $tr(P_s^T P_j) = 0$, $tr(P_s^T P_j) = 0$, $j = 1, 2, \ldots, s-1$. We can verify similarly to (6) and (7) that $tr(R_{s+1}^T R_s) = 0$, and $tr(P_{s+1}^T P_s) = 0$.

Now, it is enough to prove that $tr(R_{s+1}^T R_j) = 0$, and $tr(P_{s+1}^T P_j) = 0$.

In fact, when $j = 1$, noting that the assumptions and Lemma 2.1 implies that

$$\begin{aligned} tr(R_{s+1}^T R_1) &= tr(R_s^T R_1) - \frac{\|R_s\|^2}{\|P_s\|^2} tr(M P_s N^T)^T R_1) \\ &= -\frac{\|R_s\|^2}{\|P_s\|^2} tr(M P_s N^T)^T R_1) \\ &= -\frac{\|R_s\|^2}{\|P_s\|^2} tr(P_s^T M^T R_1 N) \\ &= -\frac{\|R_s\|^2}{\|P_s\|^2} tr(P_s^T P_1) = 0, \end{aligned}$$

and

$$\begin{aligned} tr(P_{s+1}^T P_1) &= tr\left[\left(\frac{M^T R_{s+1} N + P M^T R_{s+1} N P}{2} + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} P_s\right)^T P_1\right] \\ &= tr[R_{s+1}^T (M P_1 N^T)] + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} tr(P_s^T P_1) \\ &= \frac{\|P_1\|^2}{\|R_1\|^2} tr[R_{s+1}^T (R_1 - R_2)] \\ &= 0. \end{aligned} \tag{8}$$

Moreover, when $2 \leq j \leq k$, we obtain from Lemma 2.1 that
$tr(R_{s+1}^T R_j) = tr(R_s^T R_j) - \frac{\|R_s\|^2}{\|P_s\|^2} tr(P_s^T P_j) + \frac{\|R_s\|^2 \|R_j\|^2}{\|P_s\|^2 \|R_{j-1}\|^2} tr(P_s^T P_{j-1}) = 0.$
Similar to the proof of (8), we get $tr(P_{s+1}^T P_j) = 0$.

Hence, the conclusion holds for all positive integers k. The proof is completed.

Lemma 2.3. Suppose that $\bar{Z}$ is an arbitrary solution of Problem A, then for the sequences $\{Z_k\}$, $\{P_k\}$ generated by Algorithm 2.1, we have that

$$tr[(\bar{Z} - Z_k)^T P_k] = \| R_k \|^2, \;\; k = 1, 2, \ldots . \tag{9}$$

Proof. When k=1, From the Algorithm 2.1 and Lemma 2.2, noting that $Z_k \in CSR_P^{l \times l}$, we can obtain

$$\begin{aligned} tr[(\bar{Z} - Z_1)^T P_1] &= \frac{1}{2} tr[(\bar{Z} - Z_1)^T (M^T R_1 N + P M^T R_1 N P)] \\ &= \frac{1}{2} tr[N^T R_1^T M (\bar{Z} - Z_1) + P N^T R_1^T M P (\bar{Z} - Z_1)] \\ &= tr[R_1^T M (\bar{Z} - Z_1) N^T] \\ &= tr(R_1^T (F - M X_1 N^T)] \\ &= \|R_1\|^2. \end{aligned}$$

Assume that equality (9) holds for $k = s$, then

$$\begin{aligned} tr[(\bar{Z} - Z_{s+1})^T P_s] &= tr\left[\left(\bar{Z} - Z_s - \frac{\|R_s\|^2}{\|P_s\|^2} P_s\right)^T P_s\right] \\ &= tr[(\bar{Z} - Z_s)^T P_s] - \frac{\|R_s\|^2}{\|P_s\|^2} tr(P_s^T P_s) \\ &= \|R_s\|^2 - \|R_s\|^2 \\ &= 0. \end{aligned}$$

Moreover,

$$\begin{aligned} tr[(\bar{Z} - Z_{s+1})^T P_{s+1}] &= tr\left[(\bar{Z} - Z_{s+1})^T \left(\frac{M^T R_{s+1} N + PM^T R_{s+1} NP}{2} + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} P_s\right)\right] \\ &= \frac{1}{2} tr[N^T R_{s+1}^T M(\bar{Z} - Z_{s+1}) + PN^T R_{s+1}^T MP(\bar{Z} - Z_{s+1})] \\ &\quad + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} tr\left[(\bar{Z} - Z_{s+1})^T P_s\right] \\ &= tr[R_{s+1}^T M(\bar{Z} - Z_{s+1})N^T] \\ &= tr[R_{s+1}^T (F - MX_{s+1}N^T)] \\ &= \|R_{s+1}\|^2. \end{aligned}$$

Therefore, we complete the proof by the principle of induction.

Theorem 2.1. If Problem A is consistent, for arbitrary initial iterative matrix $Z_1 \in CSR_P^{l\times l}$, we can obtain a solution of matrix equation (1) within finite iteration steps.

Proof. Suppose that $mn < l^2$. Then, when Problem A is consistent (so is Problem I), we can obtain its solution at most $mn + 1$ iterative steps. In fact, if $R_i \neq 0$, $i = 1, 2, \ldots, mn$, Lemma 2.3 implies that $P_i \neq 0$, we can compute Z_{mn+1}, R_{mn+1} by Algorithm 2.1, which satisfy that $tr(R_{mn+1}^T R_i) = 0$, $tr(R_i^T R_j) = 0$, where $i, j = 1, 2, \ldots, mn$, $i \neq j$. Hence, the sequence $\{R_i\}$ consists of an orthogonal basis of matrix space $R^{m\times n}$, and we gain $R_{mn+1} = 0$, i.e., Z_{mn+1} is a solution of Problem A. Furthermore, from Remark 1(b), the solution of Problem I can also be obtained.

From Theorem 2.1, we deduce the following conclusion.

Corollary 1. Problem A is inconsistent (Problem I, either) if and only if there exists a positive integer k, such that $R_k \neq 0$ but $P_k = 0$.

Lemma 2.4.[8] Suppose that the consistent linear equations $My = b$ has a solution $y_0 \in R(M^T)$, then y_0 is the least-norm solution of $My = b$.

Theorem 2.2. Suppose that Problem A is consistent, let initial iterative matrix $Z_1 = M^T HN + PM^T HNP$, where arbitrary $H \in R^{m\times n}$, or especially, $Z_1 = 0 \in R^{l\times l}$, by the iterative algorithm, we can obtain the unique least-norm solution Z^* to Problem A within finite iteration steps. Moreover, the least-norm solution dual (X^*, Y^*) of Problem I can be presented by $UZ^*U^T = \begin{pmatrix} X^* & \\ & Y^* \end{pmatrix}$.

Proof. From the Algorithm 2.1 and Theorem 2.1, let initial iterative matrix $Z_1 = M^T HN + PM^T HNP$, where H is an arbitrary matrix in $R^{m\times n}$, then a solution Z^* of Problem A can be obtained within finite iterative steps, which takes on in the form of $Z^* = M^T GN + PM^T GNP$. Hence, it is enough to prove that Z^* is the least-norm of Problem A.

Consider the following matrix equations with $Z \in CSR_P^{l\times l}$

$$\begin{cases} MZN^T = F, \\ MPZPN^T = F. \end{cases}$$

Obviously, the solvability of which is equivalent to that of matrix equation (3).

Denote $vec(Z^*) = z^*$, $vec(Z) = z$, $vec(F) = f$, $vec(H) = h$, then above matrix equations can be transformed equivalently as

$$\begin{pmatrix} N \otimes M \\ (NP) \otimes (MP) \end{pmatrix} z = \begin{pmatrix} f \\ f \end{pmatrix}. \tag{10}$$

In addition, by the notations, Z^* can be rewritten as

$$z^* = \begin{pmatrix} N \otimes M \\ (NP) \otimes (MP) \end{pmatrix}^T \begin{pmatrix} h \\ h \end{pmatrix} \in R\left(\begin{pmatrix} N \otimes M \\ (NP) \otimes (MP) \end{pmatrix}^T\right),$$

which implies from Lemma 2.4 that z^* is the least-norm solution of the linear systems (10), moreover, Z^* is that of matrix equation (3). Furthermore, from the analysis in section I, the matrices product UZ^*U^T must be block-diagonal with two blocks, e.g. X^*, Y^*, and they consists of the least-norm solution pair (X^*, Y^*) of Problem I.

§3. The solution of Problem II

Suppose that the solution set of Problem I is not empty. For given matrix (X_0, Y_0) in Problem II, that is, given $Z_0 = U^T \begin{pmatrix} X_0 & \\ & Y_0 \end{pmatrix} U \in CSR_P^{l\times l}$ in Problem B, if $(X, Y) \in S_E$ (i.e., $Z = U^T \begin{pmatrix} X & \\ & Y \end{pmatrix} U \in S$), we have that

$$AXB^T + CYD^T = F \Leftrightarrow MZN^T = F \Leftrightarrow M(Z - Z_0)N^T = (F - MZ_0N^T) \Leftrightarrow M\tilde{Z}N^T = \tilde{F} \tag{11}$$

In the last equation, we denote $\tilde{Z} = Z - Z_0$ and $\tilde{F} = F - MZ_0N^T$.

If we choose initial iterative matrix $\tilde{Z}_1 = M^T\tilde{H}N + PM^T\tilde{H}NP$, where $\tilde{H} \in R^{m\times n}$ is arbitrary, or especially, let $\tilde{Z}_1 = 0 \in R^{l\times l}$, then the unique least-norm solution $\tilde{Z}^*$ of matrix equation (11) can be obtained by Algorithm 2.1. Furthermore, Theorem 2.2 illuminates that the solution dual $(\hat{X}, \hat{Y})$ of Problem II can be obtained by

$$U(\tilde{Z}^* + Z_0)U^T = \begin{pmatrix} \hat{X} & \\ & \hat{Y} \end{pmatrix}. \tag{12}$$

§4. Numerical examples

In this section, some numerical examples tested by $MATLAB$ software will be given to illustrate our results.

Example 4.1. Input matrices A, B, C, D, F, U, for instance,

$$A = \begin{pmatrix} 28 & 17 & 24 & 13 & 14 & 21 \\ 15 & 21 & 15 & 28 & 26 & 20 \\ 12 & 14 & 20 & 15 & 11 & 22 \\ 22 & 13 & 11 & 16 & 12 & 19 \\ 20 & 30 & 13 & 17 & 10 & 18 \\ 10 & 32 & 14 & 22 & 17 & 12 \end{pmatrix}, \quad B = \begin{pmatrix} 20 & 27 & 11 & 13 & 12 & 11 \\ 25 & 24 & 15 & 18 & 16 & 10 \\ 10 & 13 & 20 & 25 & 11 & 22 \\ 22 & 14 & 11 & 16 & 17 & 19 \\ 10 & 20 & 13 & 27 & 10 & 15 \\ 16 & 31 & 14 & 23 & 27 & 10 \end{pmatrix},$$

$$C = \begin{pmatrix} 10 & 11 & 11 & 13 & 12 & 11 \\ 25 & 14 & 15 & 18 & 15 & 10 \\ 10 & 13 & 10 & 15 & 11 & 13 \\ 12 & 14 & 11 & 16 & 17 & 17 \\ 10 & 10 & 13 & 17 & 10 & 15 \\ 16 & 31 & 14 & 12 & 17 & 11 \end{pmatrix}, \quad D = \begin{pmatrix} 25 & 27 & 10 & 10 & 22 & 11 \\ 21 & 24 & 11 & 18 & 30 & 20 \\ 12 & 19 & 17 & 10 & 11 & 12 \\ 22 & 15 & 12 & 16 & 18 & 10 \\ 28 & 12 & 11 & 10 & 10 & 14 \\ 16 & 11 & 12 & 18 & 17 & 20 \end{pmatrix},$$

$$F = \begin{pmatrix} 35 & 27 & 18 & 16 & 12 & 10 \\ 20 & 24 & 10 & 13 & 50 & 26 \\ 42 & 19 & 37 & 21 & 35 & 22 \\ 28 & 22 & 42 & 26 & 18 & 24 \\ 18 & 10 & 11 & 10 & 23 & 16 \\ 26 & 10 & 12 & 32 & 33 & 22 \end{pmatrix}.$$

In view of the sizes of above matrices, choose arbitrary matrix in $OR^{12\times 12}$, for example,

$$U = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then let $p = 6$ in Lemma 1.1, the symmetric orthogonal matrix is $P = U^T diag(I_6, -I_6)U$.

However, R_i (i=1,2,...) will unequal to zero for the influence of calculation errors in the iterative process. Therefore, for arbitrary positive number ε small enough, e.g., $\varepsilon = 1.0e-010$, the iteration stops whenever $\|R_k\| < \varepsilon$, and Z_k is regarded as a solution of Problem A. Meanwhile, we know from Theorem 2.2 that (X_k, Y_k) is a solution pair of Problem I.

Let $Z_1 = 0$, by Algorithm 2.1 and 102 ($< 12 \times 12$) iteration steps, we can get the solution of Problem I, that is,

$$X_{102} = \begin{pmatrix} 0.0566 & -0.0335 & 0.0756 & -0.0361 & 0.0385 & -0.0566 \\ 0.0164 & -0.0343 & -0.0893 & 0.0661 & -0.0100 & -0.0378 \\ 0.0120 & 0.1044 & -0.0010 & -0.0720 & -0.0385 & 0.0230 \\ -0.0041 & 0.1070 & -0.0404 & 0.0356 & 0.0630 & -0.0459 \\ 0.0075 & -0.1075 & 0.0049 & 0.0468 & -0.1084 & -0.0479 \\ -0.0419 & 0.0151 & 0.0503 & 0.0136 & -0.0736 & 0.0692 \end{pmatrix},$$

$$Y_{102} = \begin{pmatrix} 0.0506 & -0.0671 & -0.0326 & 0.0232 & 0.0250 & 0.0623 \\ 0.0222 & 0.0942 & -0.0144 & 0.1981 & -0.1183 & -0.1316 \\ 0.0357 & -0.0246 & -0.0116 & 0.0200 & -0.0054 & 0.0132 \\ 0.0267 & 0.0071 & 0.0194 & -0.0276 & -0.0348 & -0.0021 \\ -0.0910 & 0.0720 & 0.0143 & 0.1255 & -0.0327 & -0.0315 \\ -0.0562 & 0.0676 & 0.0430 & 0.0412 & -0.0717 & -0.0509 \end{pmatrix},$$

and $\|R_{102}\| = 9.9412e - 011 < \varepsilon$.

In this case, the convergent curve can be protracted (see Figure 1) for the Frobenius norm of the residual $(MZN^T - F)$.

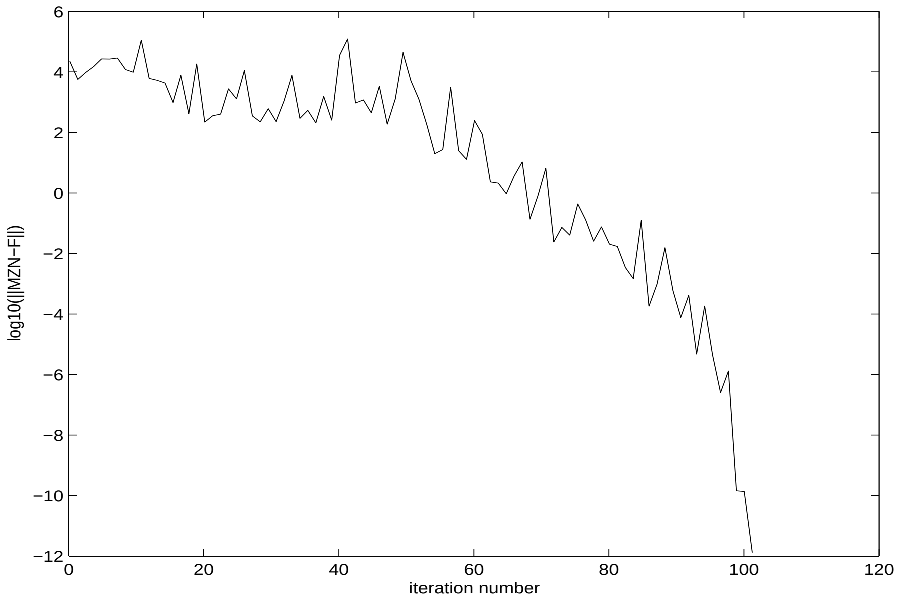

Figure 1. Convergence curve for the Frobenius norm of the residual.

In addition, for given matrices X_0 and Y_0 in Problem II,

$$X_0 = \begin{pmatrix} 4 & -3 & 7 & -2 & 9 & -8 \\ -2 & 0 & -3 & 4 & 2 & -3 \\ 8 & -6 & 4 & 2 & -5 & 4 \\ -6 & -8 & -5 & -9 & 8 & -7 \\ 9 & 8 & 2 & -4 & -6 & 6 \\ -8 & -7 & -3 & -5 & 7 & -5 \end{pmatrix}, \; Y_0 = \begin{pmatrix} -4 & 6 & -3 & 2 & -8 & -9 \\ 8 & -2 & 1 & -9 & 2 & 5 \\ -3 & 0 & -4 & 3 & 8 & -2 \\ 5 & -7 & 6 & -2 & -8 & -7 \\ 2 & -5 & -8 & -9 & -2 & 5 \\ -7 & 2 & -3 & -4 & -6 & -9 \end{pmatrix},$$

let $Z_0 = U^T \begin{pmatrix} X_0 & 0 \\ 0 & Y_0 \end{pmatrix} U$, Writing $\tilde{Z} = Z - Z_0$, and $\tilde{F} = F - MZ_0N^T$. Applying Algorithm 2.1 to matrix equation (11), we obtain its least-norm solution

$$\tilde{Z}^* = \left(\begin{array}{cccccccccccc}
-5.8153 & -0.2782 & 4.7702 & 0 & -0.8438 & -3.6153 & 0 & 0 & -5.9629 & 0 & 0 & 0 \\
2.3393 & 2.7852 & 2.7168 & 0 & 2.0293 & 3.8334 & 0 & 0 & 2.0185 & 0 & 0 & 0 \\
2.0181 & 1.5966 & -4.6136 & 0 & -7.8829 & -3.3454 & 0 & 0 & -0.7883 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.8932 & 0 & 0 & 1.3495 & -1.5618 & 0 & -1.8001 & 1.5674 & -1.8388 \\
3.5404 & 2.8399 & -2.4268 & 0 & 0.9133 & -3.6829 & 0 & 0 & -1.7462 & 0 & 0 & 0 \\
2.5268 & 2.1881 & -1.1341 & 0 & -2.4191 & 4.6972 & 0 & 0 & -1.3801 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.3810 & 0 & 0 & -0.6582 & 0.6536 & 0 & -3.1742 & 0.3432 & -1.2151 \\
0 & 0 & 0 & 5.5758 & 0 & 0 & -4.3037 & -0.9233 & 0 & -1.1176 & 6.6026 & 7.3582 \\
-3.4825 & -2.8830 & 0.0799 & 0 & 1.3228 & -0.2273 & 0 & 0 & -0.7266 & 0 & 0 & 0 \\
0 & 0 & 0 & -2.0380 & 0 & 0 & 3.6177 & 2.1986 & 0 & -4.7265 & -0.3212 & -6.2395 \\
0 & 0 & 0 & -1.2799 & 0 & 0 & -1.9036 & 1.5179 & 0 & 4.1862 & -3.2106 & 4.2403 \\
0 & 0 & 0 & 3.0779 & 0 & 0 & 0.6987 & -1.9882 & 0 & -4.0467 & 3.9708 & -1.6224
\end{array}\right),$$

Hence, it follows from equality (12) that the unique optimal solution pair $(\hat{X}, \hat{Y})$ of Problem II can be expressed as

$$\hat{X} = \begin{pmatrix}
-1.8153 & -3.8438 & 2.2298 & -1.7218 & 3.0371 & -4.3847 \\
1.5404 & 0.9133 & -0.5732 & 1.1601 & 0.2538 & 0.6829 \\
5.9819 & 1.8829 & -0.6136 & 3.5966 & -4.2117 & 0.6546 \\
-8.3393 & -10.0293 & -2.2832 & -6.2148 & 5.9815 & -3.1666 \\
5.5175 & 9.3228 & 1.9201 & -1.1170 & -6.7266 & 6.2273 \\
-10.5268 & -4.5809 & -4.1341 & -2.8119 & 8.3801 & -0.3028
\end{pmatrix},$$

$$\hat{Y} = \begin{pmatrix} -4.6582 & 5.3464 & -1.6190 & 0.7849 & -11.1742 & -9.3432 \\ 12.3037 & -2.9233 & -4.5758 & -16.3582 & 3.1176 & 11.6026 \\ -1.6505 & 1.5618 & -3.1068 & 1.1612 & 6.1999 & -3.5674 \\ 5.6987 & -5.0118 & 9.0779 & -3.6224 & -12.0467 & -10.9708 \\ 5.6177 & -7.1986 & -10.0380 & -15.2395 & -6.7265 & 5.3212 \\ -5.0964 & 3.5179 & -1.7201 & -8.2403 & -10.1862 & -12.2106 \end{pmatrix}.$$

Example 4.2. Let M and N be a 100×100 matrix with element $M_{ij} = -\frac{1}{(i+j+1)}$, $(i, j = 1, 2, ..., 100)$ and a 100×100 tri-diagonal matrix with diagonal elements equal to 4 and off-diagonal elements equal to $(-1, -\frac{1}{2}, ..., -\frac{1}{99})$ and $(1, \frac{1}{2}, ..., \frac{1}{99})$, respectively, and P be a matrix with second-diagonal elements equal to 1 and all other elements equal to 0. Let $F = M\check{Z}N^T$, where $\check{Z}$ be 100×100 matrix with all elements 1. Then the matrix equation (3) is consistent since the above matrix $\check{Z}$ is one of its solutions. Using Algorithm 2.1 and iterating 350 ($\ll 10000$) steps, we paint the convergence curve in Figure 2 for the Frobenius norm of the residual $(MZN^T - F)$.

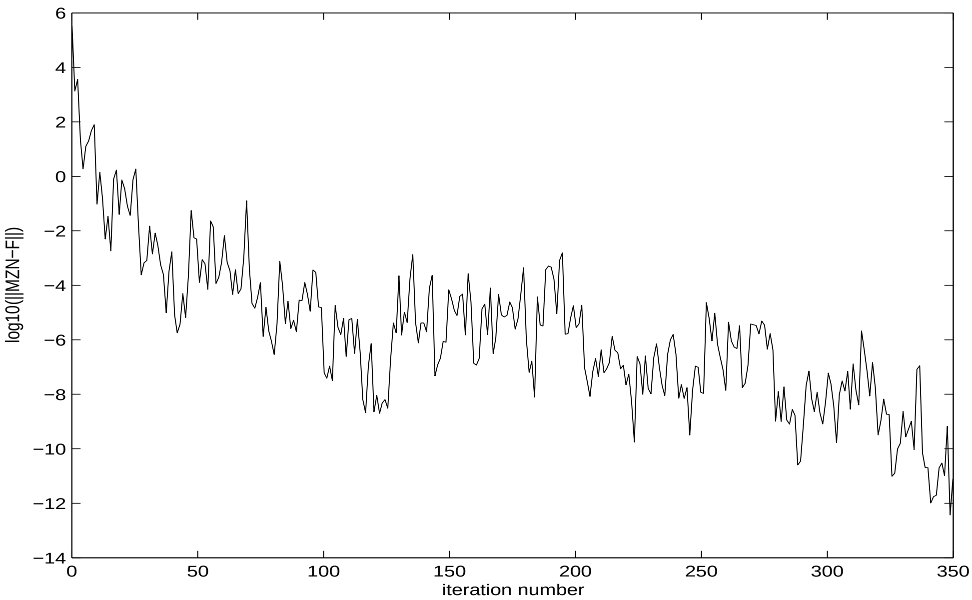

Figure 2. Convergence curve for the Frobenius norm of the residual.

From the convergency curve in Figure 2, we can see that the iterative algorithm in this paper is efficient.

The following example subjects to the conclusion in Corollary 1.

Example 4.3. Input matrices B, D, F, U, P as in Example 4.1, let $A = C = ones(6)$, and denote $M = (A,\ C)U,\ N = (B,\ D)U$.

Making use of Algorithm 2.1 solves Problem I. Let initial iterative matrix $Z_1 = 0 \in R^{6\times 6}$, we get from the 7th iteration that

$$\|R_7\| = 5.0321e + 003 \gg \|P_7\| = 2.9759.$$

From Corollary I, we know that matrix equation (1) is inconsistent.

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Smarandachely antipodal signed digraphs

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Abstract A *Smarandachely k-signed digraph* (*Smarandachely k-marked digraph*) is an ordered pair $S = (D, \sigma)$ $(S = (D, \mu))$ where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed digraph or Smarandachely 2-marked digraph is called abbreviated a *signed digraph* or a *marked digraph*. In this paper, we define the Smarandachely antipodal signed digraph $\overrightarrow{A}(D)$ of a given signed digraph $S = (D, \sigma)$ and offer a structural characterization of antipodal signed digraphs. Further, we characterize signed digraphs S for which $S \sim \overrightarrow{A}(S)$ and $\overline{S} \sim \overrightarrow{A}(S)$ where $\sim$ denotes switching equivalence and $\overrightarrow{A}(S)$ and $\overline{S}$ are denotes the Smarandachely antipodal signed digraph and complementary signed digraph of S respectively.

Keywords Smarandachely k-signed digraphs, Smarandachely k-marked digraphs, balance, switching, Smarandachely antipodal signed digraphs, negation.

§1. Introduction

For standard terminology and notion in digraph theory, we refer the reader to the classic text-books of Bondy and Murty [1] and Harary et al.[3]; the non-standard will be given in this paper as and when required.

A *Smarandachely k-signed digraph* (*Smarandachely k-marked digraph*) is an ordered pair $S = (D, \sigma)$ $(S = (D, \mu))$ where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma :$ $\mathcal{A} \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed digraph or Smarandachely 2-marked digraph is called abbreviated a *signed digraph* or a *marked digraph*. A *signed digraph* is an ordered pair $S = (D, \sigma)$, where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \to \{+, -\}$ is a function. A *marking* of S is a function $\mu : V(D) \to \{+, -\}$. A signed digraph S together with a marking μ is denoted by S_μ. A signed digraph $S = (D, \sigma)$ is *balanced* if every semicycle of S is positive (Harary et al.[3]). Equivalently, a signed digraph is balanced if every semicycle has an even number of negative arcs. The following characterization of balanced signed digraphs is obtained by E. Sampathkumar et al.[5].

Proposition 1.1.[5] A signed digraph $S = (D, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each arc $\overrightarrow{uv}$ in S satisfies $\sigma(\overrightarrow{uv}) = \mu(u)\mu(v)$.

Let $S = (D, \sigma)$ be a signed digraph. Consider the marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the signs on the arcs incident at v. *Complement* of S is a signed digraph $\overline{S} = (\overline{D}, \sigma')$, where for any arc $e = \overrightarrow{uv} \in \overline{D}$, $\sigma'(\overrightarrow{uv}) = \mu(u)\mu(v)$. Clearly, $\overline{S}$ as defined here is a balanced signed digraph due to Proposition 1.1.

In [5], the authors define switching and cycle isomorphism of a signed digraph as follows:

Let $S = (D, \sigma)$ and $S' = (D', \sigma')$, be two signed digraphs. Then S and S' are said to be *isomorphic*, if there exists an isomorphism $\phi : D \to D'$ (that is a bijection $\phi : V(D) \to V(D')$ such that if $\overrightarrow{uv}$ is an arc in D then $\overrightarrow{\phi(u)\phi(v)}$ is an arc in D') such that for any arc $\overrightarrow{e} \in D$, $\sigma(\overrightarrow{e}) = \sigma'(\phi(\overrightarrow{e}))$. For switching in signed graphs and some results involving switching refer the paper [4].

Given a marking μ of a signed digraph $S = (D, \sigma)$, *switching* S with respect to μ is the operation changing the sign of every arc $\overrightarrow{uv}$ of S' by $\mu(u)\sigma(\overrightarrow{uv})\mu(v)$. The signed digraph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called μ *switched signed digraph* or just *switched signed digraph*.

Further, a signed digraph S switches to signed digraph S' (or that they are switching equivalent to each other), written as $S \sim S'$, whenever there exists a marking of S such that $\mathcal{S}_\mu(S) \cong S'$.

Two signed digraphs $S = (D, \sigma)$ and $S' = (D', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : D \to D'$ such that the sign $\sigma(Z)$ of every semicycle Z in S equals to the sign $\sigma(\phi(Z))$ in S'.

Proposition 1.2.[4] Two signed digraphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

§2. Smarandachely antipodal signed digraphs

In [2], the authors introduced the notion antipodal digraph of a digraph as follows: For a digraph $D = (V, \mathcal{A})$, the *antipodal digraph* $\overrightarrow{A}(D)$ of $D = (V, \mathcal{A})$ is the digraph with $V(\overrightarrow{A}(D)) = V(D)$ and $\mathcal{A}(\overrightarrow{A}(D)) = \{(u, v) : u, v \in V(D) \text{ and } d_D(u, v) = diam(D)\}$.

We extend the notion of $\overrightarrow{A}(D)$ to the realm of signed digraphs. In a signed digraph $S = (D, \sigma)$, where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \to \{+, -\}$ is a function. The *Smarandachely antipodal signed digraph* $\overrightarrow{A}(S) = (\overrightarrow{A}(D), \sigma')$ of a signed digraph $S = (D, \sigma)$ is a signed digraph whose underlying digraph is $\overrightarrow{A}(D)$ called *antipodal digraph* and sign of any arc $e = \overrightarrow{uv}$ in $\overrightarrow{A}(S)$, $\sigma'(e) = \mu(u)\mu(v)$, where for any $v \in V$, $\mu(v) = \prod_{u \in N(v)} \sigma(uv)$.

Further, a signed digraph $S = (D, \sigma)$ is called *Smarandachely antipodal signed digraph*, if $S \cong \overrightarrow{A}(S')$, for some signed digraph S'. The following result indicates the limitations of the notion $\overrightarrow{A}(S)$ as introduced above, since the entire class of unbalanced signed digraphs is forbidden to be antipodal signed digraphs.

Proposition 2.1. For any signed digraph $S = (D, \sigma)$, its Smarandachely antipodal signed graph $A(S)$ is balanced.

Proof. Since sign of any arc $e = \overrightarrow{uv}$ in $\overrightarrow{A}(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S, by Proposition 1.1, $\overrightarrow{A}(S)$ is balanced.

For any positive integer k, the k^{th} iterated antipodal signed digraph $\overrightarrow{A}(S)$ of S is defined as follows:

$$\overrightarrow{A}^0(S) = S,\ A^k(S) = \overrightarrow{A}(\overrightarrow{A}^{k-1}(S)).$$

Corollary 2.2. For any signed digraph $S = (D, \sigma)$ and any positive integer k, $\overrightarrow{A}^k(S)$ is balanced.

In [2], the authors characterized those digraphs that are isomorphic to their antipodal digraphs.

Proposition 2.3.[2] For a digraph $D = (V, \mathcal{A})$, $D \cong \overrightarrow{A}(D)$ if, and only if, $D \cong K_p^*$.

Proof. First, suppose that $D \cong \overrightarrow{A}(D)$. If $(u, v) \in \mathcal{A}$ then $(u, v) \in \mathcal{A}(\overrightarrow{A}(D))$. Therefore, $d_D(u, v) = 1 = diam(D)$. Since K_p^* is the only digraph of diameter 1, we have $D \cong K_p^*$.

For the converse, if $D \cong K_p^*$, then $diam(D) = 1$ and for every pair u, v of vertices in D, the distance $d_D(u, v) = 1$. Hence, $\overrightarrow{A}(D) \cong K_p^*$ and $D \cong \overrightarrow{A}(D)$.

We now characterize the signed digraphs that are switching equivalent to their Smarandachely antipodal signed graphs.

Proposition 2.4. For any signed digraph $S = (D, \sigma)$, $S \sim \overrightarrow{A}(S)$ if, and only if, $D \cong K_p^*$ and S is balanced signed digraph.

Proof. Suppose $S \sim \overrightarrow{A}(S)$. This implies, $D \cong \overrightarrow{A}(D)$ and hence D is K_p^*. Now, if S is any signed digraph with underlying digraph as K_p^*, Proposition 2.1 implies that $\overrightarrow{A}(S)$ is balanced and hence if S is unbalanced and its $\overrightarrow{A}(S)$ being balanced can not be switching equivalent to S in accordance with Proposition 1.2. Therefore, S must be balanced.

Conversely, suppose that S is an balanced signed digraph and D is K_p^*. Then, since $\overrightarrow{A}(S)$ is balanced as per Proposition 2.1 and since $D \cong \overrightarrow{A}(D)$, the result follows from Proposition 1.2 again.

Proposition 2.5. For any two signed digraphs S and S' with the same underlying digraph, their Smarandachely antipodal signed digraphs are switching equivalent.

Proposition 2.6.[2] For a digraph $D = (V, \mathcal{A})$, $\overline{D} \cong \overrightarrow{A}(D)$ if, and only if,

i) $diam(D) = 2$.

ii) D is not strongly connected and for every pair u, v of vertices of D, the distance $d_D(u, v) = 1$ or $d_D(u, v) = \infty$.

In view of the above, we have the following result for signed digraphs:

Proposition 2.7. For any signed digraph $S = (D, \sigma)$, $\overline{S} \sim \overrightarrow{A}(S)$ if, and only if, D satisfies conditions of Proposition 2.6.

Proof. Suppose that $\overrightarrow{A}(S) \sim \overline{S}$. Then clearly we have $\overrightarrow{A}(D) \cong \overline{D}$ and hence D satisfies conditions of Proposition 2.6.

Conversely, suppose that D satisfies conditions of Proposition 2.6. Then $\overline{D} \cong \overrightarrow{A}(D)$ by Proposition 2.6. Now, if S is a signed digraph with underlying digraph satisfies conditions of Proposition 2.6, by definition of complementary signed digraph and Proposition 2.1, $\overline{S}$ and $\overrightarrow{A}(S)$ are balanced and hence, the result follows from Proposition 1.2.

The notion of *negation* $\eta(S)$ of a given signed digraph S defined in [6] as follows: $\eta(S)$ has the same underlying digraph as that of S with the sign of each arc opposite to that given to it

in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

Proposition 2.4 & 2.7 provides easy solutions to two other signed digraph switching equivalence relations, which are given in the following results.

Corollary 2.8. For any signed digraph $S = (D, \sigma)$, $S \sim \overrightarrow{A}(\eta(S))$.

Corollary 2.9. For any signed digraph $S = (D, \sigma)$, $\overline{S} \sim \overrightarrow{A}(\eta(S))$.

Problem. Characterize signed digraphs for which

i) $\eta(S) \sim \overrightarrow{A}(S)$.

ii) $\eta(\overline{S}) \sim \overrightarrow{A}(S)$.

For a signed digraph $S = (D, \sigma)$, the $\overrightarrow{A}(S)$ is balanced (Proposition 2.1). We now examine, the conditions under which negation $\eta(S)$ of $\overrightarrow{A}(S)$ is balanced.

Proposition 2.10. Let $S = (D, \sigma)$ be a signed digraph. If $\overrightarrow{A}(G)$ is bipartite then $\eta(\overrightarrow{A}(S))$ is balanced.

Proof. Since, by Proposition 2.1, $\overrightarrow{A}(S)$ is balanced, if each semicycle C in $\overrightarrow{A}(S)$ contains even number of negative arcs. Also, since $\overrightarrow{A}(D)$ is bipartite, all semicycles have even length; thus, the number of positive arcs on any semicycle C in $\overrightarrow{A}(S)$ is also even. Hence $\eta(\overrightarrow{A}(S))$ is balanced.

§3. Characterization of Smarandachely antipodal signed graphs

The following result characterize signed digraphs which are Smarandachely antipodal signed digraphs.

Proposition 3.1. A signed digraph $S = (D, \sigma)$ is a Smarandachely antipodal signed digraph if, and only if, S is balanced signed digraph and its underlying digraph D is an antipodal graph.

Proof. Suppose that S is balanced and D is a $\overrightarrow{A}(D)$. Then there exists a digraph H such that $\overrightarrow{A}(H) \cong D$. Since S is balanced, by Proposition 1.1, there exists a marking μ of D such that each arc $\overrightarrow{uv}$ in S satisfies $\sigma(\overrightarrow{uv}) = \mu(u)\mu(v)$. Now consider the signed digraph $S' = (H, \sigma')$, where for any arc e in H, $\sigma'(e)$ is the marking of the corresponding vertex in D. Then clearly, $\overrightarrow{A}(S') \cong S$. Hence S is an Smarandachely antipodal signed digraph.

Conversely, suppose that $S = (D, \sigma)$ is a Smarandachely antipodal signed digraph. Then there exists a signed digraph $S' = (H, \sigma')$ such that $\overrightarrow{A}(S') \cong S$. Hence D is the $A(D)$ of H and by Proposition 2.1, S is balanced.

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Smarandachely t-path step signed graphs

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Abstract A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$ ($\mu : V \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. E. Prisner [9] in his book Graph Dynamics defines the t-path step operator on the class of finite graphs. Given a graph G and a positive integer t, the *t-path step graph* $(G)_t$ of G is formed by taking a copy of the vertex set $V(G)$ of G, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there exists a $u - v$ path of length t in G. Analogously, one can define the *Smarandachely t-path step signed graph* $(S)_t = ((G)_t, \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $(G)_t$ called *t-path step graph* and sign of any edge $e = uv$ in $(S)_t$ is $\mu(u)\mu(v)$. It is shown that for any signed graph S, its $(S)_t$ is balanced. We then give structural characterization of Smarandachely t-path step signed graphs. Further, in this paper we characterize signed graphs which are switching equivalent to their Smarandachely 3-path step signed graphs.

Keywords Smarandachely k-signed graphs, Smarandachely k-marked graphs, signed graphs, marked graphs, balance, switching, Smarandachely t-path step signed graphs, negation.

§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [4]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$ ($\mu : V \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (Harary [3]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking μ by S_μ. Given a signed graph S one can easily define a marking μ of S as follows:

For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{u \in N(v)} \sigma(uv),$$

the marking μ of S is called *canonical* marking of S.

The following characterization of balanced signed graphs is well known.

Proposition 1.1.[6] A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.

Given a marking μ of S, by *switching* S with respect to μ we mean reversing the sign of every edge of S whenever the end vertices have opposite signs in $\mathcal{S}_\mu$[1] . We denote the signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and this signed graph is called the *μ-switched signed graph* or just *switched signed graph*. A signed graph S_1 switches to a signed graph S_2 (that is, they are *switching equivalent* to each other), written $S_1 \sim S_2$, whenever there exists a marking μ such that $\mathcal{S}_\mu(S_1) \cong S_2$.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (Sozánsky [7]) or *cycle isomorphic* (Zaslavsky [8]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known:

Proposition 1.2.[8] Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

§2. Smarandachely t-path step signed graphs

Given a graph G and a positive integer t, the *t-path step graph* $(G)_t$ of G is formed by taking a copy of the vertex set $V(G)$ of G, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there exists a $u - v$ path of length t in G. The notion of *t-path step graphs* was defined in [9], page 168.

In this paper, we shall now introduce the concept of Smarandachely t-path step signed graphs as follows: The *Smarandachely t-path step signed graph* $(S)_t = ((G)_t, \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $(G)_t$ called *t-path step graph* and sign of any edge $e = uv$ in $(S)_t$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called *Smarandachely t-path step signed graph*, if $S \cong (S')_t$, for some signed graph S'.

The following result indicates the limitations of the notion of Smarandachely t-path step signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be Smarandachely t-path step signed graphs.

Proposition 2.1. For any signed graph $S = (G, \sigma)$, its $(S)_t$ is balanced.

Proof. Since sign of any edge $e = uv$ is $(S)_t$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S, by Proposition 1.1, $(S)_t$ is balanced.

Remark. For any two signed graphs S and S' with same underlying graph, their Smarandachely t-path step signed graphs are switching equivalent.

Corollary 2.2. For any signed graph $S = (G, \sigma)$, its Smarandachely 2 (3)-path step signed graph $(S)_2$ $((S)_3)$ is balanced.

The following result characterize signed graphs which are Smarandachely t-path step signed graphs.

Proposition 2.3. A signed graph $S = (G, \sigma)$ is a Smarandachely t-path step signed graph if, and only if, S is balanced signed graph and its underlying graph G is a t-path step graph.

Proof. Suppose that S is balanced and G is a t-path step graph. Then there exists a graph H such that $(H)_t \cong G$. Since S is balanced, by Proposition 1.1, there exists a marking μ of G such that each edge $e = uv$ in S satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $(S')_t \cong S$. Hence S is a Smarandachely t-path step signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a Smarandachely t-path step signed graph. Then there exists a signed graph $S' = (H, \sigma')$ such that $(S')_t \cong S$. Hence G is the t-path step graph of H and by Proposition 2.1, S is balanced.

§3. Switching invariant Smarandachely 3-path step signed graphs

Zelinka [9] prove hat the graphs in Fig. 1 are all unicyclic graphs which are fixed in the operator $(G)_3$, i.e. graphs G such that $G \cong (G)_3$. The symbols p, q signify that the number of vertices and edges in Fig. 1.

Proposition 3.1.[9] Let G be a finite unicyclic graph such that $G \cong (G)_3$. Then either G is a circuit of length not divisible by 3, or it is some of the graphs depicted in Fig. 1.

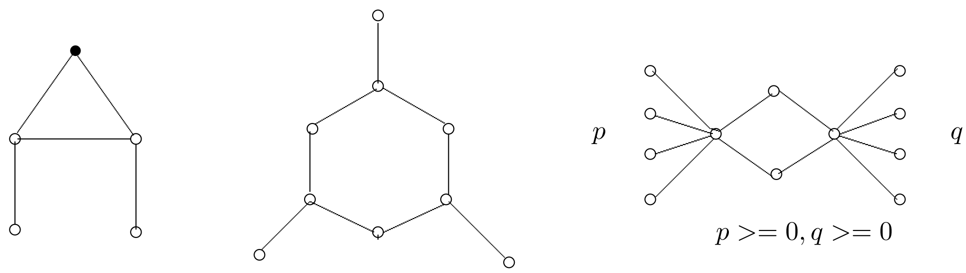

Fig.1.

In view of the above result, we have the following result for signed graphs:

Proposition 3.2. For any signed graph $S = (G, \sigma)$, $S \sim (S)_3$ if, and only if, G is a cycle of length not divisible by 3, or it is some of the graphs depicted in Fig. 1 and S is balanced.

Proof. Suppose $S \sim (S)_3$. This implies, $G \cong (G)_3$ and hence by Proposition 3.1, we see that the G must be isomorphic to either C_m, $4 \leq m \neq 3k$, k is a positive integer or the graphs depicted in Fig. 1. Now, if S is any signed graph on any of these graphs, Corollary 4 implies that $(S)_3$ is balanced and hence if S is unbalanced its Smarandachely 3-path step signed graph $(S)_3$ being balanced cannot be switching equivalent to S in accordance with Proposition 1.2. Therefore, S must be balanced.

Conversely, suppose that S is a balanced signed graph on C_m, $4 \leq m \neq 3k$, k is a positive integer or the graphs depicted in Fig. 1. Then, since $(S)_3$ is balanced as per Corollary 2.2 and since $G \cong (G)_3$ in each of these cases, the result follows from Proposition 1.2.

Problem. Characterize the signed graphs for which $S \cong (S)_3$.

The notion of *negation* $\eta(S)$ of a given signed graph S defined by Harary [3] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $(S)_t$ is balanced (Proposition 2.1). We now examine, the condition under which negation of $(S)_t$ (i.e., $\eta((S)_t)$) is balanced.

Proposition 3.3. Let $S = (G, \sigma)$ be a signed graph. If $(G)_t$ is bipartite then $\eta((S)_t)$ is balanced.

Proof. Since, by Proposition 2.1, $(S)_t$ is balanced, then every cycle in $(S)_t$ contains even number of negative edges. Also, since $(G)_t$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $(S)_t$ are also even. This implies that the same thing is true in negation of $(S)_t$. Hence $\eta((S)_t)$ is balanced.

Proposition 3.2 provides easy solutions to three other signed graph switching equivalence relations, which are given in the following results.

Corollary 3.4. For any signed graph $S = (G, \sigma)$, $\eta(S) \sim (S)_3$ if, and only if, S is unbalanced signed graph on C_{2m+1}, $m \geq 2$ or first two graphs depicted in Fig. 1.

Corollary 3.5. For any signed graph $S = (G, \sigma)$, $(\eta(S))_3 \sim (S)_3$.

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On a note of the Smarandache power function [1]

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Abstract For any positive integer n, the Smarandache power function $SP(n)$ is defined as the smallest positive integer m such that $n|m^m$, where m and n have the same prime divisors. The main purpose of this paper is to study the distribution properties of the $k-th$ power of $SP(n)$ by analytic methods, obtain three asymptotic formulas of $\sum\limits_{n\le x}(SP(n))^k$, $\sum\limits_{n\le x}\varphi((SP(n))^k)$ and $\sum\limits_{n\le x} d(SP(n))^k$ $(k>1)$, and supplement the relate conclusions in some references.

Keywords Smarandache power function, the $k-th$ power, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer n, we define the Smarandache power function $SP(n)$ as the smallest positive integer m such that $n|m^m$, where n and m have the same prime divisors. That is,

$$SP(n) = \min\left\{m : n|m^m, m \in \mathbb{N}^+, \prod_{p|m} p = \prod_{p|n} p\right\}.$$

If n runs through all natural numbers, then we can get the Smarandache power function sequence $SP(n)$: $1, 2, 3, 2, 5, 6, 7, 4, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, \cdots$, Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, denotes the factorization of n into prime powers. If $\alpha_i < p_i$, for all α_i $(i = 1, 2, \cdots, r)$, then we have $SP(n) = U(n)$, where $U(n) = \prod\limits_{p|n} p$, $\prod\limits_{p|n}$ denotes the product over all different prime divisors of n. It is clear that $SP(n)$ is not a multiplicative function.

In reference [1], Professor F. Smarandache asked us to study the properties of the sequence $SP(n)$. He has done the preliminary research about this question literature [2] $-$ [4], has obtained some important conclusions. And literature [2] has studied an average value, obtained the asymptotic formula:

$$\sum_{n\le x} SP(n) = \frac{1}{2}x^2 \prod_p \left(1 - \frac{1}{p(1+p)}\right) + O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

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Literature [3] has studied the infinite sequence astringency, has given the identical equation:

$$\sum_{n=1}^{\infty}\frac{(-1)^{\mu(n)}}{(SP(n^k))^s}=\begin{cases}\dfrac{2^s+1}{(2^s-1)\zeta(s)}, & k=1,2;\\ \dfrac{2^s+1}{(2^s-1)\zeta(s)}-\dfrac{2^s-1}{4^s}, & k=3;\\ \dfrac{2^s+1}{(2^s-1)\zeta(s)}-\dfrac{2^s-1}{4^s}+\dfrac{3^s-1}{9^s}, & k=4,5.\end{cases}$$

And literature [4] has studied the equation $SP(n^k)=\phi(n), k=1,2,3$ solubility ($\phi(n)$ is the Euler function), and has given all positive integer solution. Namely the equation $SP(n)=\phi(n)$ only has 4 positive integer solutions $n=1,4,8,18$; Equation $SP(n^3)=\phi(n)$ to have and only has 3 positive integer solutions $n=1,16,18$. In this paper, we shall use the analysis method to study the distribution properties of the $k-th$ power of $SP(n)$, gave $\sum\limits_{n\leq x}(SP(n))^k$, $\sum\limits_{n\leq x}\varphi((SP(n))^k)$ and $\sum\limits_{n\leq x}d(SP(n))^k$ $(k>1)$, some interesting asymptotic formula, has promoted the literature [2] conclusion.

Specifically as follows:

Theorem 1.1. For any random real number $x\geq 3$ and given real number k $(k>0)$, we have the asymptotic formula:

$$\sum_{n\leq x}(SP(n))^k=\frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p\left(1-\frac{1}{p^k(p+1)}\right)+O(x^{k+\frac{1}{2}+\varepsilon});$$

$$\sum_{n\leq x}\frac{(SP(n))^k}{n}=\frac{\zeta(k+1)}{k\zeta(2)}x^k\prod_p\left(1-\frac{1}{p^k(p+1)}\right)+O(x^{k+\frac{1}{2}+\varepsilon}),$$

where $\zeta(k)$ is the Riemann zeta-function, ε denotes any fixed positive number, and $\prod\limits_p$ denotes the product over all primes.

Corollary 1.1. For any random real number $x\geq 3$ and given real number $k'>0$ we have the asymptotic formula:

$$\sum_{n\leq x}(SP(n))^{\frac{1}{k'}}=\frac{6k'\zeta(\frac{1+k'}{k'})}{(k'+1)\pi^2}x^{\frac{1+k'}{k'}}\prod_p\left(1-\frac{1}{(1+p)p^{\frac{1}{k'}}}\right)+O\left(x^{\frac{k'+2}{2k'}+\varepsilon}\right).$$

Specifically, we have

$$\sum_{n\leq x}(SP(n))^{\frac{1}{2}}=\frac{4\zeta(\frac{3}{2})}{\pi^2}x^{\frac{3}{2}}\prod_p\left(1-\frac{1}{(1+p)p^{\frac{1}{2}}}\right)+O(x^{1+\varepsilon});$$

$$\sum_{n\leq x}(SP(n))^{\frac{1}{3}}=\frac{9\zeta(\frac{4}{3})}{2\pi^2}x^{\frac{4}{3}}\prod_p\left(1-\frac{1}{(1+p)p^{\frac{1}{3}}}\right)+O(x^{\frac{5}{6}+\varepsilon}).$$

Corollary 1.2. For any random real number $x\geq 3$, and $k=1,2,3$. We have the asymptotic formula:

$$\sum_{n\leq x}(SP(n))=\frac{1}{2}x^2\prod_p\left(1-\frac{1}{p(1+p)}\right)+O(x^{\frac{3}{2}+\varepsilon});$$

$$\sum_{n\leq x}(SP(n))^2=\frac{6\zeta(3)}{3\pi^2}x^3\prod_p\left(1-\frac{1}{p^2(1+p)}\right)+O(x^{\frac{5}{2}+\varepsilon});$$

$$\sum_{n\le x}(SP(n))^3=\frac{\pi^2}{60}x^4\prod_p\left(1-\frac{1}{p^3(1+p)}\right)+O(x^{\frac{7}{2}+\varepsilon}).$$

Theorem 1.2. For any random real number $x\ge 3$, we have the asymptotic formula:

$$\sum_{n\le x}\varphi\big((SP(n))^k\big)=\frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p\left(1-\frac{1}{(1+p)p^k}\right)+O(x^{k+\frac{1}{2}+\varepsilon}),$$

where $\varphi(n)$ is the Euler function

Theorem 1.3. For any random real number $x\ge 3$, we have the asymptotic formula:

$$\sum_{n\le x}d\big((SP(n))^k\big)=B_0x\ln^k x+B_1x\ln^{k-1}x+B_2x\ln^{k-2}x+\cdots+B_{k-1}x\ln x+B_kx+O(x^{\frac{1}{2}+\varepsilon}).$$

where $d(n)$ is the Dirichlet divisor function and $B_0,B_1,B_2,\cdots,B_{k-1},\ B_k$ is computable constant.

§2. Lemmas and proofs

Suppose $s=\sigma+it$ and let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}, U(n)=\prod\limits_{p|n}p$. Before the proofs of the theorem, the following Lemmas will be useful.

Lemma 2.1. For any random real number $x\ge 3$ and given real number $k\ge 1$, we have the asymptotic formula:

$$\sum_{n\le x}(U(n))^k=\frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p\left(1-\frac{1}{(1+p)p^k}\right)+O(x^{k+\frac{1}{2}+\varepsilon}).$$

Proof. Let Dirichlet's series

$$A(s)=\sum_{n=1}^{\infty}\frac{(U(n))^k}{n^s},$$

for any real number $s>1$, it is clear that $A(s)$ is absolutely convergent. Because $U(n)$ is the multiplicative function, if $\sigma>k+1$, so from the Euler's product formula [5] we have

$$\begin{aligned}A(s)&=\sum_{n=1}^{\infty}\frac{(U(n))^k}{n^s}\\&=\prod_p\left(\sum_{m=0}^{\infty}\frac{(U(p^m))^k}{p^{ms}}\right)\\&=\prod_p\left(1+\frac{p^k}{p^s}+\frac{p^k}{p^{2s}}+\cdots\right)\\&=\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}\prod_p\left(1-\frac{1}{p^k(1+p^{s-k})}\right),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. Letting $R(k)=\prod\limits_p\left(1-\frac{1}{p^k(1+p^{s-k})}\right)$. If $\sigma>k+1,\ |U(n)|\le n,\ |\sum\limits_{n=1}^{\infty}\frac{(U(n))^k}{n^\sigma}|<\zeta(\sigma-k)$.

Therefore by Perron's formula [5] with $a(n) = (U(n))^k$, $s_0 = 0$, $b = k + \frac{3}{2}$, $T = x^{k+\frac{1}{2}}$, $H(x) = x$, $B(\sigma) = \zeta(\sigma - k)$, then we have

$$\sum_{n\le x}(U(n))^k = \frac{1}{2\pi i}\int_{k+\frac{1}{2}-iT}^{k+\frac{3}{2}+iT} \frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}\mathrm{d}s + O(x^{k+\frac{1}{2}+\varepsilon}),$$

where $h(k) = \prod\limits_p \left(1 - \frac{1}{p^k(1+p)}\right)$.

To estimate the main term

$$\frac{1}{2\pi i}\int_{k+\frac{1}{2}-iT}^{k+\frac{3}{2}+iT} \frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}\mathrm{d}s,$$

we move the integral line from $s = k + \frac{3}{2} \pm iT$ to $k + \frac{1}{2} \pm iT$, then the function

$$\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}$$

have a first-order pole point at $s = k + 1$ with residue

$$\begin{aligned}
L(x) &= \underset{s=k+1}{Res}\left(\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\right)\\
&= \lim_{s\to k+1}\left((s-k-1)\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}\right)\\
&= \frac{\zeta(k+1)}{(k+1)\zeta(s)}x^{k+1}h(k).
\end{aligned}$$

Taking $T = x^{k+\frac{1}{2}}$, we can easily get the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{k+\frac{1}{2}+iT}^{k+\frac{3}{2}+iT} + \int_{k+\frac{1}{2}-iT}^{k+\frac{3}{2}+iT}\right)\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}\mathrm{d}s\right| \ll \frac{x^{2k+1}}{T} = x^{k+\frac{1}{2}},$$

$$\left|\frac{1}{2\pi i}\int_{k+\frac{1}{2}-iT}^{k+\frac{1}{2}+iT} \frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}\mathrm{d}s\right| \ll x^{k+\frac{1}{2}+\varepsilon}.$$

We may immediately obtain the asymptotic formula

$$\sum_{n\le x}(U(n))^k = \frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p\left(1 - \frac{1}{(1+p)p^k}\right) + O(x^{k+\frac{1}{2}+\varepsilon}),$$

this completes the proof of the Lemma 2.1.

Lemma 2.2. For any random real number $x \ge 3$ and given real number $k \ge 1$, and positive integer α, then we have

$$\sum_{\substack{p^\alpha\le x\\ \alpha>p}}(\alpha p)^k \ll \ln^{2k+2} x.$$

Proof. Because $\alpha > p$, so $p^p < p^\alpha \le x$, then

$$p < \frac{\ln x}{\ln p} < \ln x,\ \ \alpha \le \frac{\ln x}{\ln p},$$

also, $\sum\limits_{n\leq x} n^k = \dfrac{x^{k+1}}{k+1} + O(x^k)$. Thus,

$$\sum_{\substack{p^\alpha \leq x\\ \alpha>p}} (\alpha p)^k = \sum_{p\leq \ln x} p^k \sum_{\alpha\leq \frac{\ln x}{\ln p}} \alpha^k \ll \ln^{k+1} x \sum_{p\leq \ln x} \frac{p^k}{\ln^{k+1} p} \ll \ln^{k+1} x \sum_{p\leq \ln x} p^k.$$

Considering $\pi(x) = \sum\limits_{p\leq x} 1$, by virtue of [5], $\pi(x) = \dfrac{x}{\ln x} + O\left(\dfrac{x}{\ln^2 x}\right)$. we can get from the Able

$$\sum_{p\leq x} p^k = \pi(x)x^k - k\int_2^x \pi(t)t^{k-1}\mathrm{d}t.$$

Therefore

$$\sum_{p\leq \ln x} p^k = \frac{\ln^k x}{(k+1)} + O(\ln^{k-1} x) - k\int_2^{\ln x} \frac{t^k}{\ln t}\mathrm{d}t + O\bigg(\int_2^{\ln x} \frac{t^k}{\ln^2 t}\mathrm{d}t\bigg) = \frac{\ln^k x}{k+1} + O(\ln^{k-1} x).$$

Thus

$$\sum_{\substack{p^\alpha \leq x\\ \alpha>p}} (\alpha p)^k = \sum_{p\leq \ln x} p^k \sum_{\alpha\leq \frac{\ln x}{\ln p}} \alpha^k \ll \ln^{k+1} x \sum_{p\leq \ln x} \frac{p^k}{\ln^{k+1} p} \ll \ln^{k+1} x \sum_{p\leq \ln x} p^k \ll \ln^{2k+2} x.$$

This completes the proof of the Lemma 2.2.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem.

Proof of Theorem 1.1. Let $A = \left\{n|n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}, \alpha_i \leq p_i, i = 1, 2, \cdots, r\right\}$. When $n \in A : SP(n) = U(n)$; When $n \in \mathbb{N}^+ : SP(n) \geq U(n)$, thus

$$\sum_{n\leq x}(SP(n))^k - \sum_{n\leq x}(U(n))^k = \sum_{n\leq x}\left[(SP(n))^k - (U(n))^k\right] \ll \sum_{\substack{n\leq x\\ SP(n)>U(n)}} (SP(n))^k.$$

By the [2] known, there is integer α and prime numbers p, so $SP(n) < \alpha p$, then we can get according to Lemma 2.2

$$\sum_{\substack{n\leq x\\ SP(n)>U(n)}} (SP(n))^k < \sum_{\substack{n\leq x\\ SP(n)>U(n)}} (\alpha p)^k \ll \sum_{n\leq x}\sum_{\substack{p^\alpha<x\\ \alpha>p}} \ll x\ln^{2k+2} x.$$

Therefore

$$\sum_{n\leq x}(SP(n))^k - \sum_{n\leq x}(U(n))^k \ll x\ln^{2k+2} x.$$

From the Lemma 2.1 we have

$$\begin{aligned}\sum_{n\leq x}(SP(n))^k &= \frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p(1 - \frac{1}{p^k(1+p)}) + O(x^{k+\frac{1}{2}+\varepsilon}) + O(x\ln^{2k+1} x)\\ &= \frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p(1 - \frac{1}{p^k(1+p)}) + O(x^{k+\frac{1}{2}+\varepsilon}).\end{aligned}$$

This proves Theorem 1.1.

Proof of Corollary. According to Theorem 1.1, taking $k = \frac{1}{k'}$ the Corollary 1.1 can be obtained. Take $k = 1, 2, 3$, and $\zeta(2) = \dfrac{\pi^2}{6}, \zeta(4) = \dfrac{\pi^4}{90}$, we can achieve Corollary 1.2. Obviously so is theorem [2].

Using the similar method to complete the proofs of Theorem 1.2 and Theorem 1.3.

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Almost super Fibonacci graceful labeling

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Abstract The concept of fibonacci graceful labeling and super fibonacci graceful labeling was introduced by Kathiresan and Amutha. Let G be a $(p, q)-$ graph. An injective function $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$, where F_{q+1} is the $(q+1)^{th}$ fibonacci number, is said to be almost super fibonacci graceful graphs if the induced edge labeling $f^*(uv) = |f(u)-f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$ or $\{F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$. In this paper, we show that some well known graphs namely Path, Comb, subdivision of $(B_{2,n} : w_i)$, $1 \leq i \leq n$ and some special types of extension of cycle related graphs are almost super fibonacci graceful labeling.

Keywords Graceful labeling, Fibonacci graceful labeling, almost super fibonacci graceful labeling.

§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A path of length n is denoted by P_{n+1}. A cycle of length n is denoted by C_n. G^+ is a graph obtained from the graph G by attaching pendant vertex to each vertex of G. Graph labelings, where the vertices are assigned certain values subject to some conditions, have often motivated by practical problems. In the last five decades enormous work has been done on this subject [2]. The concept of graceful labeling was first introduced by Rosa [6] in 1967.

A function f is a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. The slightly stronger concept of almost graceful was introduced by moulton [5]. A function f is an almost graceful labeling of a graph G if the vertex labels are come from $\{0, 1, 2, \ldots, q-1, q+1\}$ while the edge labels are $1, 2, \ldots, q-1, q$ or $1, 2, \ldots, q-1, q+1$. The notion of Fibonacci graceful labeling and Super Fibonacci graceful labeling was introduced by Kathiresan and Amutha [4]. We call a function f, a fibonacci graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, F_q\}$, where F_q is the q^{th} fibonacci number of the fibonacci series $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$, such that each edge uv is assigned the labels $|f(u) - f(v)|$,

the resulting edge labels are $F_1, F_2, \ldots, F_q$. Also, we call super fibonacci graceful labeling, An injective function $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$, where F_q is the q^{th} fibonacci number, is said to be a super fibonacci graceful labeling if the induced edge labeling $|f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$. In the labeling problems the induced labelings must be distinct. So to introduce fibonacci graceful labelings we assume $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots,$ as the sequence of fibonacci numbers instead of $0, 1, 2, \ldots,$ [3] these concepts motivate us to define the following.

An injective function $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$, where F_{q+1} is the $(q+1)^{th}$ fibonacci number, is said to be almost super fibonacci graceful if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$. Frucht [1] has introduced a stronger version of almost graceful graphs by permitting as vertex labels $\{0, 1, 2, \ldots, q-1, q+1\}$ and as edge labels $\{1, 2, \ldots, q\}$. He calls such a labeling Pseudo graceful. In a stronger version of almost super fibonacci graceful graphs by permitting as vertex labels $\{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ and an edge labels $\{F_1, F_2, \ldots, F_q\}$. Then such labeling is called pseudo fibonacci graceful labeling. In this paper, we show that some well known graphs namely Path, Comb, subdivision of $(B_{2,n} : w_i)$, $1 \leq i \leq n$ and some special types of extension of cycle related graphs are almost super fibonacci graceful graph.

§2. Main results

In this section, we show that some well known graphs and some extension of cycle related graphs are almost super fibonacci graceful graph.

Definition 2.1. Let G be a (p, q) graph. An injective function $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$, where F_{q+1} is the $(q+1)^{th}$ fibonacci number, is said to be almost super fibonacci graceful graphs if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$ or $\{F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$.

The following theorem shows that the graph P_n is an almost super fibonacci graceful graph.

Theorem 2.1. The path P_n is an almost super fibonacci graceful graph.

Proof. Let P_n be a path of n vertices. Let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set and $\{e_1, e_2, \ldots, e_{n-1}\}$ be the edge set, where $e_i = u_i u_{i+1}$, $1 \leq i \leq n$.

Case(i): n is odd.

Define $f : V(P_n) \to \{F_0, F_1, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u_i) = F_{n-2(i+1)}$, $1 \leq i \leq \frac{n-3}{2}$, $f(u_i) = F_{2(i+1)-(n-1)}, \frac{n-3}{2} + 1 \leq i \leq n-3$, $f(u_{n-2}) = F_0$, $f(u_{n-1}) = F_{n-2}$, $f(u_n) = F_n$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(u_i u_{i+1}) : 1 \leq i \leq \frac{n-3}{2}\}$.

Then

$$\begin{aligned} E_1 &= \{|f(u_i) - f(u_{i+1})| : 1 \leq i \leq \frac{n-3}{2}\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{\frac{n-3}{2}-1}) - f(u_{\frac{n-3}{2}})|, \\ &\quad |f(u_{\frac{n-3}{2}}) - f(u_{\frac{n-3}{2}+1})|\} \\ &= \{|F_{n-4} - F_{n-6}|, |F_{n-6} - F_{n-8}|, \ldots, |F_3 - F_1|, |F_1 - F_2|\} \\ &= \{F_{n-5}, F_{n-7}, \ldots, F_2, F_1\}. \end{aligned}$$

Let $E_2 = \{f^*(u_iu_{i+1}) : \frac{n-3}{2} + 1 \leq i \leq n-3\}$.
Then

$$\begin{aligned} E_2 &= \left\{|f(u_i) - f(u_{i+1})| : \frac{n-3}{2} + 1 \leq i \leq n-3\right\} \\ &= \Big\{|f(u_{\frac{n-3}{2}+1}) - f(u_{\frac{n-3}{2}+2})|, |f(u_{\frac{n-3}{2}+2}) - f(u_{\frac{n-3}{2}+3})|, \\ &\quad |f(u_{n-4}) - f(u_{n-3})|, |f(u_{n-3}) - f(u_{n-2})|\Big\} \\ &= \{|F_2 - F_4|, |F_4 - F_6|, \ldots, |F_{n-5} - F_{n-3}|, |F_{n-3} - F_0|\} \\ &= \{F_3, F_5, \ldots, F_{n-4}, F_{n-3}\}. \end{aligned}$$

Let $E_3 = \{f^*(u_{n-2}u_{n-1}), f^*(u_{n-1}u_n)\}$.
Then

$$\begin{aligned} E_3 &= \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|F_0 - F_{n-2}|, |F_{n-2} - F_n|\} \\ &= \{F_{n-2}, F_{n-1}\}. \end{aligned}$$

Therefore

$$\begin{aligned} E &= E_1 \cup E_2 \cup E_3 \\ &= \{F_1, F_2, \ldots, F_{n-1}\}. \end{aligned}$$

Thus, the edge labels are distinct. Therefore, P_n admits almost super fibonacci graceful labeling. Hence, P_n is an almost super fibonacci graceful graph.

For example the almost super fibonacci graceful labeling of P_5 is shown in Fig. 1.

P_5 :

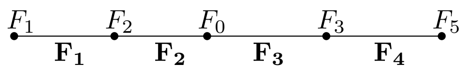

Fig.1

Case(ii) : n is even.

Define $f : V(P_n) \rightarrow \{F_0, F_1, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u_i) = F_{n-4-2(i-1)}$, $1 \leq i \leq \frac{n-4}{2}$, $f(u_i) = F_{2(i+1)-n+1}$, $\frac{n-4}{2} + 1 \leq i \leq n-3$, $f(u_{n-2}) = F_0$, $f(u_{n-1}) = F_{n-2}$, $f(u_n) = F_n$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(u_iu_{i+1}) : 1 \leq i \leq \frac{n-4}{2}\}$.
Then

$$\begin{aligned} E_1 &= \{|f(u_i) - f(u_{i+1})| : 1 \leq i \leq \frac{n-4}{2}\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{\frac{n-4}{2}-1}) - f(u_{\frac{n-4}{2}})|, \\ &\quad |f(u_{\frac{n-4}{2}}) - f(u_{\frac{n-4}{2}+1})|\} \\ &= \{|F_{n-4} - F_{n-6}|, |F_{n-6} - F_{n-8}|, \ldots, |F_4 - F_2|, |F_2 - F_1|\} \\ &= \{F_{n-5}, F_{n-7}, \ldots, F_3, F_1\}. \end{aligned}$$

Let $E_2 = \{f^*(u_iu_{i+1}) : \frac{n-4}{2} + 1 \le i \le n-4\}$.
Then

$$\begin{aligned}
E_2 &= \{|f(u_i) - f(u_{i+1})| : \frac{n-4}{2} + 1 \le i \le n-4\} \\
&= \{|f(u_{\frac{n-4}{2}+1}) - f(u_{\frac{n-4}{2}+2})|, |f(u_{\frac{n-4}{2}+2}) - f(u_{\frac{n-4}{2}+3})|, \ldots, \\
&\quad |f(u_{n-5}) - f(u_{n-4})|, |f(u_{n-4}) - f(u_{n-3})|\} \\
&= \{|F_1 - F_3|, |F_3 - F_5|, \ldots, |F_{n-7} - F_{n-5}|, |F_{n-5} - F_{n-3}|\} \\
&= \{F_2, F_4, \ldots, F_{n-6}, F_{n-4}\}.
\end{aligned}$$

Let $E_3 = \{f^*(u_{n-3}u_{n-2}), f^*(u_{n-2}u_{n-1}), f^*(u_{n-1}u_n)\}$.
Then

$$\begin{aligned}
E_3 &= \{|f(u_{n-3}) - f(u_{n-2})|, |f(u_{n-2}) - f(u_{n-1})|, \\
&\quad |f(u_{n-1}) - f(u_n)|\} \\
&= \{|F_{n-3} - F_0|, |F_0 - F_{n-2}|, |F_{n-2} - F_n|\} \\
&= \{F_{n-3}, F_{n-2}, F_{n-1}\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
E &= E_1 \cup E_2 \cup E_3 \\
&= \{F_1, F_2, \ldots, F_{n-1}\}.
\end{aligned}$$

Thus, the edge labels are distinct. Therefore, P_n admits almost super fibonacci graceful labeling. Hence, P_n is an almost super fibonacci graceful graph.

For example the almost super fibonacci graceful labeling of P_6 is shown in Fig. 2.

P_6 :

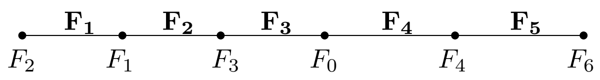

Fig.2.

Definition 2.2. The graph obtained by joining a single pendant edge to each vertex of path is called a comb and is denoted by $P_n \odot K_1$ or P_n^+.

Theorem 2.2. The path P_n^+ is an almost super fibonacci graceful graph.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of path P_n and $v_1, v_2, \ldots, v_n$ be the vertices adjacent to each vertex of P_n. Also, $|V(G)| = 2n$ and $|E(G)| = 2n-1$. Define $f : V(P_n \odot K_1) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u_i) = F_{2i}$, $1 \le i \le n-2$, $f(v_i) = F_{2i-1}$, $1 \le i \le n-1$, $f(u_{n-1}) = F_0$, $f(u_n) = F_{2n-2}$, $f(v_n) = F_{2n}$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(u_iu_{i+1}) : i = 1, 2, \ldots, n-3\}$.

Then

$$\begin{aligned}
E_1 &= \{|f(u_i) - f(u_{i+1})| : i = 1, 2, \ldots, n-3\} \\
&= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{n-4}) - f(u_{n-3})|, \\
&\quad |f(u_{n-3}) - f(u_{n-2})|\} \\
&= \{|F_2 - F_4|, |F_4 - F_6|, |F_6 - F_8|, \ldots, |F_{2n-8} - F_{2n-6}|, \\
&\quad |F_{2n-6} - F_{2n-4}|\} \\
&= \{F_3, F_5, \ldots, F_{2n-7}, F_{2n-5}\}.
\end{aligned}$$

Let $E_2 = \{f^*(u_i u_{i+1}) : n-2 \leq i \leq n-1\}$.
Then

$$\begin{aligned}
E_2 &= \{f^*(u_{n-2}u_{n-1}), f^*(u_{n-1}u_n)\} \\
&= \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\
&= \{|F_{2n-4} - F_0|, |F_0 - F_{2n-2}|\} \\
&= \{F_{2n-4}, F_{2n-2}\}.
\end{aligned}$$

Let $E_3 = \{f^*(u_1 v_1)\}$.
Then

$$\begin{aligned}
E_3 &= \{|f(u_1) - f(v_1)|\} \\
&= \{|F_2 - F_1|\} \\
&= \{F_1\}.
\end{aligned}$$

Let $E_4 = \{f^*(u_{i+1}v_{i+1}) : 1 \leq i \leq n-3\}$.
Then

$$\begin{aligned}
E_4 &= \{|f(u_{i+1}) - f(v_{i+1})| : 1 \leq i \leq n-3\} \\
&= \{|f(u_2 - f(v_2)|, |f(u_3) - f(v_3)|, \ldots, |f(u_{n-3}) - f(v_{n-3})|, \\
&\quad |f(u_{n-2}) - f(v_{n-2})|\} \\
&= \{|F_4 - F_3|, |F_6 - F_5|, \ldots, |F_{2n-6} - F_{2n-7}|, |F_{2n-4} - F_{2n-5}|\} \\
&= \{F_2, F_4, \ldots, F_{2n-8}, F_{2n-6}\}.
\end{aligned}$$

Let $E_5 = \{f^*(u_{n-1}v_{n-1}), f^*(u_n v_n)\}$.
Then

$$\begin{aligned}
E_5 &= \{|f(u_{n-1}) - f(v_{n-1})|, |f(u_n) - f(v_n)|\} \\
&= \{|F_0 - F_{2n-3}|, |F_{2n} - F_{2n-2}|\} \\
&= \{F_{2n-3}, F_{2n-1}\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
E &= E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \\
&= \{F_1, F_2, \ldots, F_{2n-1}\}.
\end{aligned}$$

Thus, the edge labels are distinct. Therefore, $P_n \odot K_1$ admits almost super fibonacci graceful labeling. Hence, $P_n \odot K_1$ is an almost super fibonacci graceful graph.

This example shows that the graph $P_5 \odot K_1$ is an almost super fibonacci graceful graph.

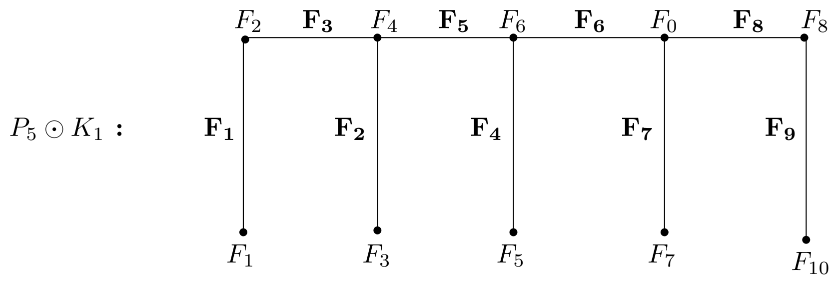

Fig.3

Definition 2.3. Let $K_{1,n}$ be a star with $n+1$ vertices ($n \geq 2$). Adjoin a pendant edge at each n pendant vertex of $K_{1,n}$. The resultant graph is called a extension of $K_{1,n}$ and is denoted by $K_{1,n}^{+}$.

Next, theorem shows that the graph $K_{1,n}^{+}$ is an almost super Fibonacci graceful graph.

Theorem 2.3. The graph $G = K_{1,n}^{+}$ is an almost super fibonacci graceful graph.

Proof. Let (V_1, V_2) be the bipartion of $K_{1,n}$, where $V_1 = \{u_0\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$ and $v_1, v_2, \ldots, v_n$ be the pendant vertices joined with $u_1, u_2, \ldots, u_n$ respectively. Also, $|V(G)| = 2n+1$ and $|E(G)| = 2n$.

Case(i) : n is odd.

Define $f : V(G) \rightarrow \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u_0) = F_0$, $f(u_{2i-1}) = F_{4i-3}$, $1 \leq i \leq \frac{n+1}{2}$, $f(u_{2i}) = F_{4i}$, $1 \leq i \leq \frac{n-1}{2}$, $f(v_{2i-1}) = F_{4i-1}$, $1 \leq i \leq \frac{n+1}{2}$, $f(v_{2i}) = F_{4i-2}$, $1 \leq i \leq \frac{n-1}{2}$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(u_0 u_{2i-1})$: $1 \leq i \leq \frac{n+1}{2}\}$.

Then

$$\begin{aligned}
E_1 &= \{|f(u_0) - f(u_{2i-1})| : 1 \leq i \leq \frac{n+1}{2}\} \\
&= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_3)|, \ldots, |f(u_0) - f(u_{n-2})|, |f(u_0) - f(u_n)|\} \\
&= \{|F_0 - F_1|, |F_0 - F_5|, \ldots, |F_0 - F_{2n-5}|, |F_0 - F_{2n-1}|\} \\
&= \{F_1, F_5, \ldots, F_{2n-5}, F_{2n-1}\}.
\end{aligned}$$

Let $E_2 = \{f^*(u_0 u_{2i})$: $1 \leq i \leq \frac{n-1}{2}\}$.

Then

$$\begin{aligned}
E_2 &= \{|f(u_0) - f(u_{2i})| : 1 \leq i \leq \frac{n-1}{2}\} \\
&= \{|f(u_0) - f(u_2)|, |f(u_0) - f(u_4)|, \ldots, |f(u_0) - f(u_{n-3})|, |f(u_0) - f(u_{n-1})|\} \\
&= \{|F_0 - F_4|, |F_0 - F_8|, \ldots, |F_0 - F_{2n-6}|, |F_0 - F_{2n-2}|\} \\
&= \{F_4, F_8, \ldots, F_{2n-6}, F_{2n-2}\}.
\end{aligned}$$

Let $E_3 = \{f^*(u_{2i-1} v_{2i-1})$: $1 \leq i \leq \frac{n+1}{2}\}$.

Then

$$\begin{aligned} E_3 &= \{|f(u_{2i-1}) - f(v_{2i-1})| : 1 \le i \le \frac{n+1}{2}\} \\ &= \{|f(u_1) - f(v_1)|, |f(u_3) - f(v_3)|, \ldots, |f(u_{n-2}) - f(v_{n-2})|, |f(u_n) - f(v_n)|\} \\ &= \{|F_1 - F_3|, |F_5 - F_7|, \ldots, |F_{2n-5} - F_{2n-3}|, |F_{2n-1} - F_{2n+1}|\} \\ &= \{F_2, F_6, \ldots, F_{2n-4}, F_{2n}\}. \end{aligned}$$

Let $E_4 = \{f^*(u_{2i}v_{2i})\colon\ 1 \le i \le \frac{n-1}{2}\}$.
Then

$$\begin{aligned} E_4 &= \{|f(u_{2i}) - f(v_{2i})| : 1 \le i \le \frac{n-1}{2}\} \\ &= \{|f(u_2) - f(v_2)|, |f(u_4) - f(v_4)|, \ldots, |f(u_{n-3}) - f(v_{n-3})|, \\ &\quad |f(u_{n-1}) - f(v_{n-1})|\} \\ &= \{|F_4 - F_2|, |F_8 - F_6|, \ldots, |F_{2n-6} - F_{2n-8}|, |F_{2n-2} - F_{2n-4}|\} \\ &= \{F_3, F_7, \ldots, F_{2n-7}, F_{2n-3}\}. \end{aligned}$$

Therefore

$$\begin{aligned} E &= E_1 \cup E_2 \cup E_3 \cup E_4 \\ &= \{F_1, F_2, \ldots, F_{2n}\}. \end{aligned}$$

Thus, the edge labels are distinct. Therefore, $K_{1,n}^{+}$ admits almost super fibonacci graceful labeling. Hence, $K_{1,n}^{+}$ is an almost super fibonacci graceful graph.

For example the almost super fibonacci graceful labeling of $K_{1,5}^{+}$ is shown in Fig. 4.

$K_{1,5}^{+}$:

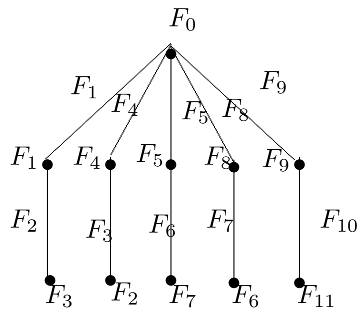

Fig.4

Case(ii) : n is even.

Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u_0) = F_0$, $f(v_1) = F_1$, $f(u_{2i-1}) = F_{4i-2}$, $1 \le i \le \frac{n}{2}$, $f(u_{2i}) = F_{4i-1}$, $1 \le i \le \frac{n}{2}$, $f(v_{2i}) = F_{4i+1}$, $1 \le i \le \frac{n}{2}$, $f(v_{2i-1}) = F_{4i-4}$, $2 \le i \le \frac{n}{2}$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(u_0u_{2i-1})\colon\ 1 \le i \le \frac{n}{2}\}$.

Then

$$\begin{aligned} E_1 &= \{|f(u_0) - f(u_{2i-1})| : 1 \leq i \leq \frac{n}{2}\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_3)|, \ldots, |f(u_0) - f(u_{n-3})|, |f(u_0) - f(u_{n-1})|\} \\ &= \{|F_0 - F_2|, |F_0 - F_6|, \ldots, |F_0 - F_{2n-6}|, |F_0 - F_{2n-2}|\} \\ &= \{F_2, F_6, \ldots, F_{2n-6}, F_{2n-2}\}. \end{aligned}$$

Let $E_2 = \{f^*(u_0u_{2i})\text{: } 1 \leq i \leq \frac{n}{2}\}$.
Then

$$\begin{aligned} E_2 &= \{|f(u_0) - f(u_{2i})| : 1 \leq i \leq \frac{n}{2}\} \\ &= \{|f(u_0) - f(u_2)|, |f(u_0) - f(u_4)|, \ldots, |f(u_0) - f(u_{n-2})|, |f(u_0) - f(u_n)|\} \\ &= \{|F_0 - F_3|, |F_0 - F_7|, \ldots, |F_0 - F_{2n-5}|, |F_0 - F_{2n-1}|\} \\ &= \{F_3, F_7, \ldots, F_{2n-5}, F_{2n-1}\}. \end{aligned}$$

Let $E_3 = \{f^*(u_1v_1)\}$.
Then

$$\begin{aligned} E_3 &= \{|f(u_1) - f(v_1)|\} \\ &= \{|F_2 - F_1|\} \\ &= \{F_1\}. \end{aligned}$$

Let $E_4 = \{f^*(u_{2i-1}v_{2i-1})\text{: } 2 \leq i \leq \frac{n}{2}\}$.
Then

$$\begin{aligned} E_4 &= \{|f(u_{2i-1}) - f(v_{2i-1})| : 2 \leq i \leq \frac{n}{2}\} \\ &= \{|f(u_3) - f(v_3)|, |f(u_5) - f(v_5)|, \ldots, |f(u_{n-3}) - f(v_{n-3})|, \\ &\quad |f(u_{n-1}) - f(v_{n-1})|\} \\ &= \{|F_6 - F_4|, |F_{10} - F_8|, \ldots, |F_{2n-6} - F_{2n-8}|, |F_{2n-2} - F_{2n-4}|\} \\ &= \{F_5, F_9, \ldots, F_{2n-7}, F_{2n-3}\}. \end{aligned}$$

Let $E_5 = \{f^*(u_{2i}v_{2i}) : 1 \leq i \leq \frac{n}{2}\}$.
Then

$$\begin{aligned} E_5 &= \{|f(u_{2i}) - f(v_{2i})| : 1 \leq i \leq \frac{n}{2}\} \\ &= \{|f(u_2) - f(v_2)|, |f(u_4) - f(v_4)|, \ldots, |f(u_{n-1}) - f(v_{n-1})|, |f(u_n) - f(v_n)|\} \\ &= \{|F_3 - F_5|, |F_7 - F_9|, \ldots, |F_{2n-5} - F_{2n-3}|, |F_{2n-1} - F_{2n+1}|\} \\ &= \{F_4, F_8, \ldots, F_{2n-4}, F_{2n}\}. \end{aligned}$$

Therefore

$$\begin{aligned} E &= E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \\ &= \{F_1, F_2, \ldots, F_{2n}\}. \end{aligned}$$

Thus, the edge labels are distinct. Therefore, $K^+_{1,n}$ admits almost super fibonacci graceful labeling. Hence, $K^+_{1,n}$ is an almost super fibonacci graceful graph.

For example the almost super fibonacci graceful labeling of $K^+_{1,4}$ is shown in Fig. 5.

$K^+_{1,4}$ **:**

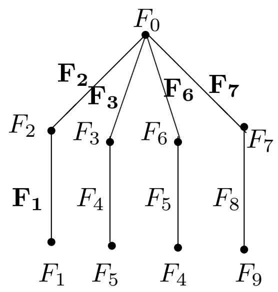

Fig.5

Definition 2.4. The graph $G = F_n \oplus P_3$ consists of a fan F_n and a path P_3 of length two which is attached with the maximum degree of the vertex of F_n.

Theorem 2.4. $G = F_n \oplus P_3$ is an almost super fibonacci graceful graph for $n \geq 3$.

Proof. Let $V(G) = U \cup V$, where $U = \{u_0, u_1, \ldots, u_n\}$ be the vertex set of F_n and $V = \{u_0 = v_1, v_2, v_3\}$ be the vertex set of P_3. Also, $|V(G)| = n+3$ and $|E(G)| = 2n+1$. Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u_0) = f(v_1) = F_0$, $f(u_i) = F_{2i-1}$, $1 \leq i \leq n$, $f(v_{i+1}) = F_{2(n+i)-2}$, $1 \leq i \leq 2$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(u_i u_{i+1})\colon i = 1, 2, \cdots, n-1\}$.

Then

$$\begin{aligned} E_1 &= \{|f(u_i) - f(u_{i+1})| : i = 1, 2, \ldots, n-1\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{n-2}) - f(u_{n-1})|, \\ &\quad |f(u_{n-1}) - f(u_n)|\} \\ &= \{|F_1 - F_3|, |F_3 - F_5|, \ldots, |F_{2n-5} - F_{2n-3}|, |F_{2n-3} - F_{2n-1}|\} \\ &= \{F_2, F_4, \ldots, F_{2n-4}, F_{2n-2}\}. \end{aligned}$$

Let $E_2 = \{f^*(u_0 u_i)\colon i = 1, 2, \cdots, n\}$.

Then

$$\begin{aligned} E_2 &= \{|f(u_0) - f(u_i)| : i = 1, 2, \ldots, n\} \\ &= \{|F_0 - F_{2i-1}| : i = 1, 2, \ldots, n\} \\ &= \{|F_0 - F_1|, |F_0 - F_3|, \ldots, |F_0 - F_{2n-3}|, |F_0 - F_{2n-1}|\} \\ &= \{F_1, F_3, \ldots, F_{2n-3}, F_{2n-1}\}. \end{aligned}$$

.

Let E_3=$\{f^*(v_i v_{i+1}) : i = 1, 2\}$.

Then

$$\begin{aligned} E_3 &= \{|f(v_i) - f(v_{i+1})| : i = 1, 2\} \\ &= \{|F_0 - F_{2n+i-1}| : i = 1, 2\} \\ &= \{|F_0 - F_{2n}|, |F_{2n} - F_{2n+2}|\} \\ &= \{F_{2n}, F_{2n+1}\}. \end{aligned}$$

Therefore

$$\begin{aligned} E &= E_1 \cup E_2 \cup E_3 \\ &= \{F_1, F_2, \ldots, F_{2n+1}\}. \end{aligned}$$

Thus, the edge labels are distinct. There, $F_n \oplus P_3$ admits almost super fibonacci graceful labeling. Hence, $F_n \oplus P_3$is an almost super fibonacci graceful graph. This example shows that the graph $F_5 \oplus P_3$ is an almost super fibonacci graceful graph.

$F_5 \oplus P_3$**:**

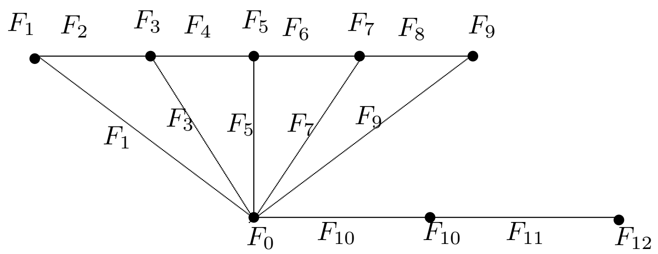

Fig.6

Definition 2.5. An $(n, 2t)$ - graph consists of a cycle of length n with two copies of t-edge path attached to two adjacent vertices and it is denoted by $C_n@2P_t$.

Next theorem shows that the $(n, 2t)$ - graph is an almost super Fibonacci graceful graph.

Theorem 2.5. The $(n, 2t)$ graph G, where $t = 1$, $n \equiv 0(\text{mod } 3)$ is an almost super fibonacci graceful graph.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of cycle of length n. Let v, w be the vertices of paths P_1 and P_2 joined with u_1, u_n respectively. Also, $|V(G)| = |E(G)| = n + 2$. Define Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u_1) = F_0$, $f(u_n) = F_{n+1}$, $f(u_{n-1}) = F_{n-1}$, $f(v) = F_1$, $f(w) = F_{n+3}$. For $l = 1, 2, \ldots, \frac{n-3}{3}$, $f(u_{i+1}) = F_{2i-3(l-1)}$, $3l - 2 \le i \le 3l$.

Next, we claim that the edge labels are distinct. We have to find the edge labeling between the vertex u_1 and starting vertex u_2 of the first loop.

Let $E_1 = \{f^*(u_1u_2)\}$.

Then

$$\begin{aligned} E_1 &= \{|f(u_1) - f(u_2)|\} \\ &= \{|F_0 - F_2|\} \\ &= \{F_2\}. \end{aligned}$$

Let $E_2 = \{f^*(u_{i+1}u_{i+2})\colon 1 \leq i \leq 2\}$.

For $l = 1$

Then

$$\begin{aligned} E_2 &= \{|f^*(u_{i+1}u_{i+2})| : 1 \leq i \leq 2\} \\ &= \{|f(u_{i+1}) - f(u_{i+2})| : 1 \leq i \leq 2\} \\ &= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \\ &= \{|F_2 - F_4|, |F_4 - F_6|\} \\ &= \{F_3, F_5\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_4 of the first loop and starting vertex u_5 of the second loop.

Let

$$\begin{aligned} E_2^1 &= \{f^*(u_4u_5)\} \\ &= \{|\ f(u_4) - f(u_5)\ |\} \\ &= \{|\ F_6 - F_5\ |\} \\ &= \{F_4\}. \end{aligned}$$

For $l = 2$

Let $E_3 = \{f^*(u_{i+1}u_{i+2})\ {:}4 \leq i \leq 5\}$.

Then

$$\begin{aligned} E_3 &= \{|f(u_{i+1} - f(u_{i+2})| : 4 \leq i \leq 5\} \\ &= \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \\ &= \{|F_5 - F_7|, |F_7 - F_9|\} \\ &= \{F_6, F_8\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_7 of the second loop and starting vertex u_8 of the third loop.

Let

$$\begin{aligned} E_3^1 &= \{f^*(u_7u_8)\} \\ &= \{|\ f(u_7) - f(u_8)\ |\} \\ &= \{|\ F_9 - F_8\ |\} \\ &= \{F_7\}. \end{aligned}$$

etc.,

For $l = \frac{n-3}{3} - 1$

Let $E_{\frac{n-3}{3}-1} = \{f^*(u_{i+1}u_{i+2}) : n - 8 \leq i \leq n - 7\}$.

Then

$$\begin{aligned} E_{\frac{n-3}{3}-1} &= \{|f(u_{i+1}) - f(u_{i+2})| : n-8 \leq i \leq n-7\} \\ &= \{|f(u_{n-7}) - f(u_{n-6})|, |f(u_{n-6}) - f(u_{n-5})|\} \\ &= \{|F_{n-7} - F_{n-5}|, |F_{n-5} - F_{n-3}|\} \\ &= \{F_{n-6}, F_{n-4}\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_{n-5} of the $(\frac{n-3}{3} - 1)^{th}$ loop and starting vertex u_{n-4} of the $(\frac{n-3}{3})^{rd}$ loop.

Let

$$\begin{aligned} E^1_{\frac{n-3}{3}-1} &= \{f^*(u_{n-5}u_{n-4})\} \\ &= \{| f(u_{n-5}) - f(u_{n-4}) |\} \\ &= \{| F_{n-3} - F_{n-4} |\} \\ &= \{F_{n-5}\}. \end{aligned}$$

For $l = \frac{n-3}{3}$

Let $E_{\frac{n-3}{3}} = \{f^*(u_{i+1}u_{i+2})$: $n-5 \leq i \leq n-4\}$.

Then

$$\begin{aligned} E_{\frac{n-3}{3}} &= \{| f(u_{i+1}) - f(u_{i+2}) |: n-5 \leq i \leq n-4\} \\ &= \{| f(u_{n-4}) - f(u_{n-3}) |, | f(u_{n-3}) - f(u_{n-2}) |\} \\ &= \{| F_{n-4} - F_{n-2} |, | F_{n-2} - F_n |\} \\ &= \{F_{n-3}, F_{n-1}\}. \end{aligned}$$

Let $E^* = \{f^*(u_{n-2}u_{n-1}), f^*(u_{n-1}u_n), f^*(u_nu_1), f^*(u_n\omega), f^*(u_1v)\}$.

Then

$$\begin{aligned} E^* &= \{| f(u_{n-2}) - f(u_{n-1}) |, | f(u_{n-1}) - f(u_n) |, | f(u_n) - f(u_1) |, \\ &\quad | f(u_n) - f(\omega) |, | f(u_1) - f(v) |\} \\ &= \{| F_n - F_{n-1} |, | F_{n-1} - F_{n+1} |, | F_{n+1} - F_0 |, | F_{n+1} - F_{n+3} |, | F_0 - F_1 |\} \\ &= \{F_{n-2}, F_n, F_{n+1}, F_{n+2}, F_1\}. \end{aligned}$$

Therefore

$$\begin{aligned} E &= (E_1 \cup E_2 \cup, \ldots, E_{\frac{n-3}{3}}) \cup (E^1_2 \cup E^1_3 \cup, \ldots, E^1_{\frac{n-3}{3}-1}) \cup E^* \\ &= \{F_1, F_2, \ldots, F_{n+1}, F_{n+2}\}. \end{aligned}$$

Thus, the edge labels are distinct. Therefore, the $(n, 2t)$-graph admits almost super fibonacci graceful labeling. Hence, the $(n, 2t)$-graph G is an almost super fibonacci graceful graph. This example shows that the graph $G = (6, 2t)$ is an almost super fibonacci graceful graph.

$G = (6, 2t)$ **:**

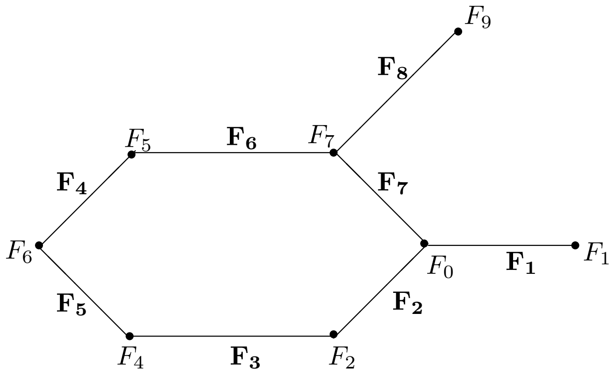

Fig.7

Definition 2.6. The graph $G = C_3@2P_n$ consists of a cycle C_3 together with the two copies of path P_n of length n, two end vertices u_1, v_1 of P_n is joined with two vertices of C_3.

Theorem 2.6. The graph $G = C_3@2P_n$ is an almost super Fibonacci graceful graph except when $n \equiv 2 \pmod 3$.

Proof. Let $V(G) = U \cup V \cup W$, where $W = \{w_1, w_2, w_3\}$ be the vertices of C_3, $U = \{w_1 = u_1, u_2, \ldots, u_{n+1}\}$ and $V = \{w_2 = v_1, v_2, \ldots, v_{n+1}\}$ be the vertex set of path P_{n+1} of length n. Also, $|V(G)| = 2n + 3 = |E(G)|$.

Case(i) : $n \equiv 1 \pmod 3$.

Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(w_3) = F_{n+4}$, when $n \equiv 1 \pmod 3$, $f(u_{n+1-(i-1)}) = F_{2(n-i+3)}$, $1 \le i \le 2$, $f(u_{n-1}) = F_0$, $f(u_{n-2}) = F_{2n+1}$, $f(u_{n-3}) = F_{2n-1}$,$f(v_i) = F_{n-i+3}$, $1 \le i \le n+1$. For $l = 1, 2, \ldots, \frac{n-4}{3}$, $f(u_{n-3-i}) = F_{2n-2(i-1)+3(l-1)}$, $3l - 2 \le i \le 3l$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(u_{n+1}u_n), f^*(u_n u_{n-1}), f^*(u_{n-1}u_{n-2}), f^*(u_{n-2}u_{n-3})\}$.

Then

$$\begin{aligned} E_1 &= \{| f(u_{n+1}) - f(u_n) |, | f(u_n) - f(u_{n-1}) |, | f(u_{n-1}) - f(u_{n-2}) |, \\ &\quad | f(u_{n-2}) - f(u_{n-3}) |\} \\ &= \{| F_{2n+4} - F_{2n+2} |, | F_{2n+2} - F_0 |, | F_0 - F_{2n+1} |, | F_{2n+1} - F_{2n-1} |\} \\ &= \{F_{2n+3}, F_{2n+2}, F_{2n+1}, F_{2n}\}. \end{aligned}$$

We have to find the edge labeling between the vertex u_{n-3} and starting vertex u_{n-4} of the first loop.

Let $E_2 = \{f^*(u_{n-3}u_{n-4})\}$

Then

$$\begin{aligned} E_2 &= \{| f(u_{n-3}) - f(u_{n-4}) |\} \\ &= \{| F_{2n-1} - F_{2n} |\} \\ &= \{F_{2n-2}\}. \end{aligned}$$

For $l = 1$

Let $E_3 = \{f^*(u_{n-3-i}u_{n-4-i}) : 1 \leq i \leq 2\}$

Then

$$\begin{aligned} E_3 &= \{\mid f(u_{n-3-i}) - f(u_{n-4-i}) \mid: 1 \leq i \leq 2\} \\ &= \{\mid f(u_{n-4}) - f(u_{n-5}) \mid, \mid f(u_{n-5}) - f(u_{n-6}) \mid\} \\ &= \{\mid F_{2n} - F_{2n-2} \mid, \mid F_{2n-2} - F_{2n-4} \mid\} \\ &= \{F_{2n-1}, F_{2n-3}\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_{n-6} of the first loop and starting vertex u_{n-7} of the second loop.

Let $E_3^1 = \{f^*(u_{n-6}u_{n-7})\}$

Then

$$\begin{aligned} E_3^1 &= \{\mid f(u_{n-6}) - f(u_{n-7}) \mid\} \\ &= \{\mid F_{2n-4} - F_{2n-3} \mid\} \\ &= \{F_{2n-5}\}. \end{aligned}$$

For $l = 2$

Let $E_4 = \{f^*(u_{n-3-i}u_{n-4-i}) : 4 \leq i \leq 5)\}$

Then

$$\begin{aligned} E_4 &= \{\mid f(u_{n-3-i}) - f(u_{n-4-i}) \mid: 4 \leq i \leq 5\} \\ &= \{\mid f(u_{n-7}) - f(u_{n-8}) \mid, \mid f(u_{n-8}) - f(u_{n-9}) \mid\} \\ &= \{\mid F_{2n-3} - F_{2n-5} \mid, \mid F_{2n-5} - F_{2n-7} \mid\} \\ &= \{F_{2n-4}, F_{2n-6}\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_{n-9} of the first loop and starting vertex u_{n-10} of the third loop.

Let $E_4^1 = \{f^*(u_{n-9}u_{n-10})\}$

Then

$$\begin{aligned} E_4^1 &= \{\mid f(u_{n-9}) - f(u_{n-10}) \mid\} \\ &= \{\mid F_{2n-7} - F_{2n-6} \mid\} \\ &= \{F_{2n-8}\}. \end{aligned}$$

etc.,

For $l = \frac{n-4}{3} - 1$

Let $E_{\frac{n-4}{3}-1} = \{f^*(u_{n-3-i}u_{n-4-i}) : n-9 \leq i \leq n-8\}$

Then

$$\begin{aligned} E_{\frac{n-4}{3}-1} &= \{\mid f(u_{n-3-i}) - f(u_{n-4-i}) \mid: n-9 \leq i \leq n-8\} \\ &= \{\mid f(u_6) - f(u_5) \mid, \mid f(u_5) - f(u_4) \mid\} \\ &= \{\mid F_{n+10} - F_{n+8} \mid, \mid F_{n+8} - F_{n+6} \mid\} \\ &= \{F_{n+9}, F_{n+7}\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_6 of the $(\frac{n-4}{3}-1)^{th}$ loop and starting vertex $(\frac{n-4}{3}-1)^{rd}$ of the third loop.

Let $E^1_{\frac{n-4}{3}-1} = \{f^*(u_4u_3)\}$

Then

$$\begin{aligned} E^1_{\frac{n-4}{3}-1} &= \{\mid f(u_4) - f(u_3) \mid\} \\ &= \{\mid F_{n+6} - F_{n+4} \mid\} \\ &= \{F_{n+5}\}. \end{aligned}$$

For $l = \frac{n-4}{3}$

Let $E_{\frac{n-4}{3}} = \{f^*(u_{n-3-i}u_{n-4-i}) : n-6 \leq i \leq n-5\}$

Then

$$\begin{aligned} E_{\frac{n-4}{3}} &= \{|f(u_{n-3-i}) - f(u_{n-4-i})| : n-6 \leq i \leq n-5\} \\ &= \{|f(u_3) - f(u_2)|, |f(u_2) - f(u_1)|\} \\ &= \{|F_{n+7} - F_{n+5}|, |F_{n+5} - F_{n+3}|\} \\ &= \{F_{n+6}, F_{n+4}\}. \end{aligned}$$

Let $E^*_1 = \{f^*(u_{n+1-(i-1)}u_{n-(i-1)}) : i = 1\}$

Then

$$\begin{aligned} E^*_1 &= \{|f(u_{n+1-(i-1)}) - f(u_{n-(i-1)})| : i = 1\} \\ &= \{|f(u_{n+1}) - f(u_n)|\} \\ &= \{|F_{2n+4} - F_{2n+2}|\} \\ &= \{F_{2n+3}\}. \end{aligned}$$

Let $E^*_2 = \{f^*(\omega_1\omega_3), f^*(\omega_2\omega_3), f^*(\omega_1\omega_2)\}$

Then

$$\begin{aligned} E^*_2 &= \{|f(\omega_1) - f(\omega_3)|, |f(\omega_2) - f(\omega_3)|, |f(\omega_1) - f(\omega_2)|\} \\ &= \{|F_{n+3} - F_{n+4}|, |F_{n+2} - F_{n+4}|, |F_{n+3} - F_{n+2}|\} \\ &= \{F_{n+2}, F_{n+3}, F_{n+1}\}. \end{aligned}$$

Let $E^*_3 = \{f^*(v_iv_{i+1}) : 1 \leq i \leq n\}$.

Then

$$\begin{aligned} E^*_3 &= \{|f(v_i) - f(v_{i+1})| : 1 \leq i \leq n\} \\ &= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|, \ldots, |f(v_{n-1}) - f(v_n)|, |f(v_n) - f(v_{n+1})|\} \\ &= \{|F_{n+2} - F_{n+1}|, |F_{n+1} - F_n|, \ldots, |F_4 - F_3|, |F_3 - F_2|\} \\ &= \{F_n, F_{n-1}, \ldots, F_2, F_1\}. \end{aligned}$$

Therefore

$$\begin{aligned} E &= (E_1 \cup E_2 \cup, \ldots, E_{\frac{n-4}{3}}) \cup (E^1_3 \cup E^1_4 \cup, \ldots, \cup E^1_{\frac{n-4}{3}-1}) \cup (E^*_1 \cup E^*_2 \cup E^*_3) \\ &= \{F_1, F_2, \ldots, F_{2n+2}, F_{2n+3}\}. \end{aligned}$$

Thus, the edge labels are distinct. Therefore, the graph $G = C_3@2P_n$ admits almost super fibonacci graceful labeling. Hence, the graph $G = C_3@2P_n$ is an almost super fibonacci graceful graph.

For example the almost super fibonacci graceful labeling of $C_3@2P_{10}$ is shown in Fig. 8.

$C_3@2P_{10}$:

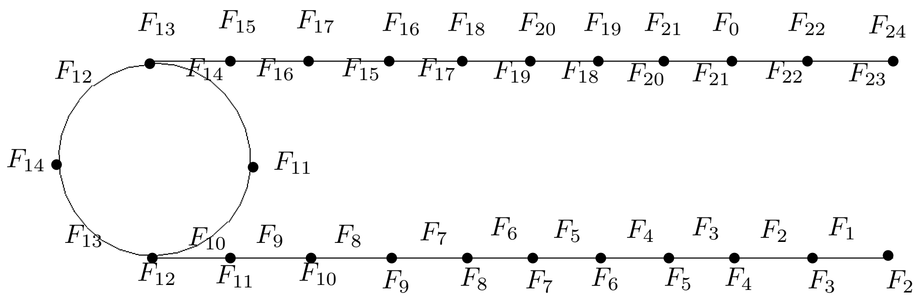

Fig.8

Case(ii): when $n \equiv 0 \pmod 3$

Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(w_3) = F_{n+3}$, when $n \equiv 0 \pmod 3$, $f(u_{n+1-(i-1)}) = F_{2(n-i+3)}$, $1 \le i \le 2$, $f(u_{n-1}) = F_0$, $f(u_{n-2}) = F_{2n+1}$, $f(u_{n-3}) = F_{2n-1}$, $f(u_2) = F_{n+6}$, $f(u_1) = F_{n+4}$, $f(v_i) = F_{n-i+3}$, $1 \le i \le n+1$. For $l = 1, 2, \ldots, \frac{n-6}{3}$, $f(u_{n-3-i}) = F_{2n-2(i-1)+3(l-1)}$, $3l - 2 \le i \le 3l$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(u_{n+1}u_n), f^*(u_n u_{n-1}), f^*(u_{n-1}u_{n-2}), f^*(u_{n-2}u_{n-3})\}$.

Then

$$\begin{aligned} E_1 &= \{| f(u_{n+1}) - f(u_n)|, |f(u_n) - f(u_{n-1})|, |f(u_{n-1}) - f(u_{n-2})|, \\ &\quad |f(u_{n-2}) - f(u_{n-3})|\} \\ &= \{| F_{2n+4} - F_{2n+2}|, |F_{2n+2} - F_0|, |F_0 - F_{2n+1}|, |F_{2n+1} - F_{2n-1}|\} \\ &= \{F_{2n+3}, F_{2n+2}, F_{2n+1}, F_{2n}\}. \end{aligned}$$

We have to find the edge labeling between the vertex u_{n-3} and starting vertex u_{n-4} of the first loop.

Let $E_2 = \{f^*(u_{n-3}u_{n-4})\}$

Then

$$\begin{aligned} E_2 &= \{|f(u_{n-3}) - f(u_{n-4})|\} \\ &= \{|F_{2n-1} - F_{2n}|\} \\ &= \{F_{2n-2}\}. \end{aligned}$$

For $l = 1$

Let $E_3 = \{f^*(u_{n-3-i}u_{n-4-i}) : 1 \le i \le 2)\}$.

Then

$$\begin{aligned} E_3 &= \{|f(u_{n-3-i}) - f(u_{n-4-i})| : 1 \leq i \leq 2\} \\ &= \{|f(u_{n-4}) - f(u_{n-5})|, |f(u_{n-5}) - f(u_{n-6})|\} \\ &= \{|F_{2n} - F_{2n-2}|, |F_{2n-2} - F_{2n-4}|\} \\ &= \{F_{2n-1}, F_{2n-3}\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_{n-6} of the first loop and starting vertex u_{n-7} of the second loop.

Let $E_3^1 = \{f^*(u_{n-6}u_{n-7})\}$

Then

$$\begin{aligned} E_3^1 &= \{|f(u_{n-6}) - f(u_{n-7})|\} \\ &= \{|F_{2n-4} - F_{2n-3}|\} \\ &= \{F_{2n-5}\}. \end{aligned}$$

For $l = 2$

Let $E_4 = \{f^*(u_{n-3-i}u_{n-4-i}) : 4 \leq i \leq 5)\}$.

Then

$$\begin{aligned} E_4 &= \{|f(u_{n-3-i}) - f(u_{n-4-i})| : 4 \leq i \leq 5\} \\ &= \{|f(u_{n-7}) - f(u_{n-8})|, |f(u_{n-8}) - f(u_{n-9})|\} \\ &= \{|F_{2n-3} - F_{2n-5}|, |F_{2n-5} - F_{2n-7}|\} \\ &= \{F_{2n-4}, F_{2n-6}\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_{n-9} of the second loop and starting vertex u_{n-10} of the third loop.

Let $E_4^1 = \{f^*(u_{n-9}u_{n-10})\}$

Then

$$\begin{aligned} E_4^1 &= \{|f(u_{n-9}) - f(u_{n-10})|\} \\ &= \{|F_{2n-7} - F_{2n-6}|\} \\ &= \{F_{2n-8}\}. \end{aligned}$$

etc.

For $l = \frac{n-6}{3} - 1$

Let $E_{\frac{n-6}{3}-1} = \{f^*(u_{n-3-i}u_{n-4-i}) : n-11 \leq i \leq n-10\}$.

Then

$$\begin{aligned} E_{\frac{n-6}{3}-1} &= \{|f(u_{n-3-i}) - f(u_{n-4-i})| : n-11 \leq i \leq n-10\} \\ &= \{|f(u_8) - f(u_7)|, |f(u_7) - f(u_6)|\} \\ &= \{|F_{n+12} - F_{n+10}|, |F_{n+10} - F_{n+8}|\} \\ &= \{F_{n+11}, F_{n+9}\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_6 of the $(\frac{n-6}{3}-1)^{th}$ loop and starting vertex u_5 of the $(\frac{n-6}{3})^{rd}$ loop.

Let $E^1_{\frac{n-6}{3}-1} = \{f^*(u_6u_5)\}$.

Then

$$\begin{aligned} E^1_{\frac{n-6}{3}-1} &= \{|f(u_6) - f(u_5)|\} \\ &= \{|F_{n+8} - F_{n+9}|\} \\ &= \{F_{n+7}\}. \end{aligned}$$

For $l = \frac{n-6}{3}$

Let $E_{\frac{n-6}{3}} = \{f^*(u_{n-3-i}u_{n-4-i}) : n-8 \leq i \leq n-7\}$.

Then

$$\begin{aligned} E_{\frac{n-6}{3}} &= \{|f(u_{n-3-i}) - f(u_{n-4-i})| : n-8 \leq i \leq n-7\} \\ &= \{|f(u_5) - f(u_4)|, |f(u_4) - f(u_3)|\} \\ &= \{|F_{n+9} - F_{n+7}|, |F_{n+7} - F_{n+5}|\} \\ &= \{F_{n+8}, F_{n+6}\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_3 of the $(\frac{n-6}{3})^{th}$ loop and the vertex u_2.

Let

$$\begin{aligned} E^*_1 &= \{f^*(u_3u_2)\} \\ &= \{|f(u_3) - f(u_2)|\} \\ &= \{|F_{n+5} - F_{n+6}|\} \\ &= \{F_{n+4}\}. \end{aligned}$$

Let $E^*_2 = \{f^*(u_2u_1), f^*(\omega_1\omega_3), f^*(\omega_2\omega_3), f^*(\omega_1\omega_2)\}$.

Then

$$\begin{aligned} E^*_2 &= \{|f(u_2) - f(u_1)|, |f(\omega_1) - f(\omega_3)|, |f(\omega_2) - f(\omega_3)|, |f(\omega_1) - f(\omega_2)|\} \\ &= \{|F_{n+6} - F_{n+4}|, |F_{n+4} - F_{n+3}|, |F_{n+2} - F_{n+3}|, |F_{n+4} - F_{n+2}|\} \\ &= \{F_{n+5}, F_{n+2}, F_{n+1}, F_{n+3}\}. \end{aligned}$$

Let $E^*_3 = \{f^*(v_iv_{i+1}) : 1 \leq i \leq n\}$.

Then

$$\begin{aligned} E^*_3 &= \{|f(v_i) - f(v_{i+1})| : 1 \leq i \leq n\} \\ &= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|, \ldots, |f(v_{n-1}) - f(v_n)|, |f(v_n) - f(v_{n+1})|\} \\ &= \{|F_{n+2} - F_{n+1}|, |F_{n+1} - F_n|, \ldots, |F_4 - F_3|, |F_3 - F_2|\} \\ &= \{F_n, F_{n-1}, \ldots, F_2, F_1\}. \end{aligned}$$

Therefore

$$\begin{aligned} E &= (E_1 \cup E_2 \cup, \ldots, E_{\frac{n-6}{3}}) \cup (E^1_3 \cup E^1_4 \cup, \ldots, E^1_{\frac{n-6}{3}-1}) \cup (E^*_1 \cup E^*_2 \cup E^*_3) \\ &= \{F_1, F_2, \ldots, F_{2n+2}, F_{2n+3}\}. \end{aligned}$$

Thus, the edge labels are distinct. Therefore, the graph $G = C_3@2P_n$ admits almost super fibonacci graceful labeling. Hence, the graph $G = C_3@2P_n$ is an almost super fibonacci graceful graph.

For example the almost super fibonacci graceful labeling of $C_3@2P_9$ is shown in Fig.9.

$C_3@2P_9$:

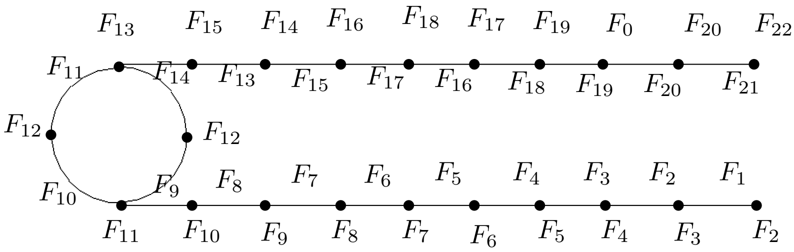

Fig.9

Definition 2.7. The graph $G = C_{3n+1} \ominus K_{1,2}$ consists of a cycle C_{3n+1} of length $3n + 1$ and $K_{1,2}$ is attached with the vertex u_n of C_{3n+1}.

The following theorem shows that the graph $G = C_{3n+1} \ominus K_{1,2}$ is an almost super fibonacci graceful graph.

Theorem 2.7. The graph $G = C_{3n+1} \ominus K_{1,2}$ is an almost super fibonacci graceful graph.

Proof. Let $V(G) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of C_{3n+1} and $V_2 = \{v, w\}$ be the end vertices of $K_{1,2}$. Also, $|V(G)| = |E(G)| = n + 2$.

Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u_i) = F_{2(i-1)}$, $1 \le i \le 2$, $f(u_i) = F_{2i-3}$, $3 \le i \le 4$, $f(v) = F_n$, $f(w) = F_{n+3}$. For $l = 1, 2, \ldots, \frac{n-4}{3}$, $f(u_{i+4}) = F_{2i-3l+5}$, $3l - 2 \le i \le 3l$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(u_i u_{i+1}) : 1 \le i \le 3\}$.

Then

$$
\begin{aligned}
E_1 &= \{|f(u_i) - f(u_{i+1})| : 1 \le i \le 3\} \\
&= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \\
&= \{|F_0 - F_2|, |F_2 - F_3|, |F_3 - F_5|\} \\
&= \{F_2, F_1, F_4\}.
\end{aligned}
$$

We have to find the edge labeling between the vertex u_4 and starting vertex u_5 of the first loop.

Let $E_2 = \{f^*(u_4 u_5)\}$.

Then

$$
\begin{aligned}
E_2 &= \{|f(u_4) - f(u_5)|\} \\
&= \{|F_5 - F_4|\} \\
&= \{F_3\}.
\end{aligned}
$$

For $l = 1$
Let $E_3 = \{f^*(u_{i+4}u_{i+5}) : 1 \leq i \leq 2\}$.
Then

$$\begin{aligned} E_3 &= \{|f(u_{i+4}) - f(u_{i+5})| : 1 \leq i \leq 2\} \\ &= \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \\ &= \{|F_4 - F_6|, |F_6 - F_8|\} \\ &= \{F_5, F_7\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_7 of the first loop and starting vertex u_8 of the second loop.
Let $E_3^{(1)} = \{f^*(u_7u_8)\}$.
Then

$$\begin{aligned} E_3^{(1)} &= \{|f(u_7) - f(u_8)|\} \\ &= \{|F_8 - F_7|\} \\ &= \{F_6\}. \end{aligned}$$

For $l = 2$
Let $E_4 = \{f^*(u_{i+4}u_{i+5}) : 4 \leq i \leq 5\}$.
Then

$$\begin{aligned} E_4 &= \{|f(u_{i+4}) - f(u_{i+5})| : 4 \leq i \leq 5\} \\ &= \{|f(u_8) - f(u_9)|, |f(u_9) - f(u_{10})|\} \\ &= \{|F_7 - F_9|, |F_9 - F_{11}|\} \\ &= \{F_8, F_{10}\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_{10} of the second loop and starting vertex u_{11} of the third loop.
Let $E_4^{(1)} = \{f^*(u_{10}u_{11})\}$.
Then

$$\begin{aligned} E_4^{(1)} &= \{|f(u_{10}) - f(u_{11})|\} \\ &= \{|F_{11} - F_{10}|\} \\ &= \{F_9\}. \end{aligned}$$

etc.
For $l = \frac{n-4}{3} - 1$
Let $E_{\frac{n-4}{3}-1} = \{f^*(u_{i+4}u_{i+5}) : n-9 \leq i \leq n-8\}$.
Then

$$\begin{aligned} E_{\frac{n-4}{3}-1} &= \{|f(u_{i+4}) - f(u_{i+5})| : n-9 \leq i \leq n-8\} \\ &= \{|f(u_{n-5}) - f(u_{n-4})|, |f(u_{n-4}) - f(u_{n-3})|\} \\ &= \{|F_{n-6} - F_{n-4}|, |F_{n-4} - F_{n-2}|\} \\ &= \{F_{n-5}, F_{n-3}\}. \end{aligned}$$

We have to find the edge labeling between the end vertex u_{n-3} of the $(\frac{n-4}{3}-1)^{th}$ loop and starting vertex u_{n-2} of the $(\frac{n-4}{3})^{rd}$ loop.

Let $E^{(1)}_{\frac{n-4}{3}-1} = \{f^*(u_{n-3}u_{n-2})\}$.

Then

$$\begin{aligned} E^{(1)}_{\frac{n-4}{3}-1} &= \{|f(u_{n-3}) - f(u_{n-2})|\} \\ &= \{|F_{n-2} - F_{n-3}|\} \\ &= \{F_{n-4}\}. \end{aligned}$$

For $l = \frac{n-4}{3}$

Let $E_{\frac{n-4}{3}} = \{f^*(u_{i+4}u_{i+5}) : n-6 \leq i \leq n-5\}$.

Then

$$\begin{aligned} E_{\frac{n-4}{3}} &= \{|f(u_{i+4}) - f(u_{i+5})| : n-6 \leq i \leq n-5\} \\ &= \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|F_{n-3} - F_{n-1}|, |F_{n-1} - F_{n+1}|\} \\ &= \{F_{n-2}, F_n\}. \end{aligned}$$

Let $E^* = \{f^*(u_n u_1), f^*(u_n v), f^*(u_n w)\}$.

Then

$$\begin{aligned} E^* &= \{|f(u_n) - f(u_1)|, |f(u_n) - f(v)|, |f(u_n) - f(w)|\} \\ &= \{|F_{n+1} - F_0|, |F_{n+1} - F_n|, |F_{n+1} - F_{n+3}|\} \\ &= \{F_{n+1}, F_{n-1}, F_{n+2}\}. \end{aligned}$$

Theorefore

$$\begin{aligned} E &= (E_1 \cup E_2, \ldots, \cup E_{\frac{n-4}{3}}) \cup (E^{(1)}_3 \cup E^{(1)}_4 \cup, \ldots, \cup E^{(1)}_{\frac{n-4}{3}-1}) \cup E^* \\ &= \{F_1, F_2, \ldots, F_{n+1}, F_{n+2}\}. \end{aligned}$$

Thus, the edge labels are distinct. Therefore, $C_{3n+1} \ominus K_{1,2}$ admits almost super fibonacci graceful labeling. Hence, $C_{3n+1} \ominus K_{1,2}$ is an almost super fibonacci graceful graph.

This example shows that the graph $C_7 \ominus K_{1,2}$ is an almost super fibonacci graceful graph.

$C_7 \ominus K_{1,2}$:

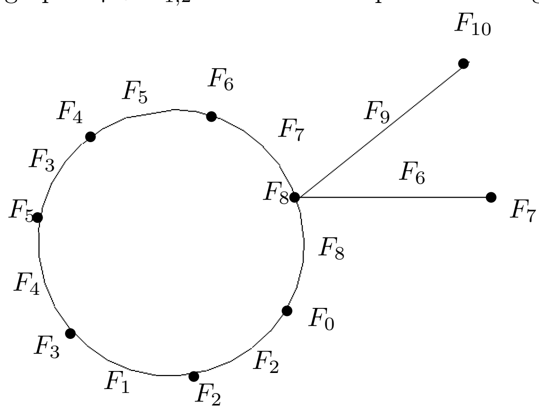

Fig.10

Definition 2.8. $G = K_{1,n} \oslash K_{1,2}$ is a graph in which $K_{1,2}$ is joined with each pendant vertex of $K_{1,n}$.

Theorem 2.8. The graph $G = K_{1,n} \oslash K_{1,2}$ is an almost super Fibonacci graceful graph.

Proof. Let $\{u_0, u_1, u_2, \ldots, u_n\}$ be the vertex set of $K_{1,n}$ and $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ be the vertices joined with the pendant vertices $u_1, u_2, \ldots, u_n$ of $K_{1,n}$ respectively. Also, $|V(G)| = 3n + 1$ and $|E(G)| = 3n$.

Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u_0) = F_0$, $f(u_i) = F_{3i-1}$, $1 \le i \le n$, $f(v_i) = F_{3i}$, $1 \le i \le n$, $f(w_i) = F_{3i+1}$, $1 \le i \le n$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(u_0 u_i) : i = 1, 2, \ldots, n\}$.

Then

$$\begin{aligned}
E_1 &= \{|f(u_0) - f(u_i)| : i = 1, 2, \ldots n\} \\
&= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, \ldots, |f(u_0) - f(u_{n-1})|, |f(u_0) - f(u_n)|\} \\
&= \{|F_0 - F_2|, |F_0 - F_5|, \ldots, |F_0 - F_{3n-4}|, |F_0 - F_{3n-1}|\} \\
&= \{F_2, F_5, \ldots, F_{3n-4}, F_{3n-1}\}.
\end{aligned}$$

Let $E_2 = \{f^*(u_i v_i) : i = 1, 2, \ldots n\}$.

Then

$$\begin{aligned}
E_2 &= \{|f(u_i) - f(v_i)| : i = 1, 2, \ldots, n\} \\
&= \{|f(u_1) - f(v_1)|, |f(u_2) - f(v_2)|, \ldots, |f(u_{n-1}) - f(v_{n-1})|, |f(u_n) - f(v_n)|\} \\
&= \{|F_2 - F_3|, |F_5 - F_6|, \ldots, |F_{3n-4} - F_{3n-3}|, |F_{3n-1} - F_{3n}|\} \\
&= \{F_1, F_4, \ldots, F_{3n-5}, F_{3n-2}\}.
\end{aligned}$$

Let $E_3 = \{f^*(u_i w_i) : i = 1, 2, \ldots n\}$.

Then

$$\begin{aligned}
E_3 &= \{|f(u_i) - f(w_i)| : i = 1, 2, \ldots, n\} \\
&= \{|f(u_1) - f(w_1)|, |f(u_2) - f(w_2)|, \ldots, |f(u_{n-1}) - f(w_{n-1})|, |f(u_n) - f(w_n)|\} \\
&= \{|F_2 - F_4|, |F_5 - F_7|, \ldots, |F_{3n-4} - F_{3n-2}|, |F_{3n-1} - F_{3n+1}|\} \\
&= \{F_3, F_6, \ldots, F_{3n-3}, F_{3n}\}.
\end{aligned}$$

Theorefore

$$\begin{aligned}
E &= E_1 \cup E_2 \cup E_3 \\
&= \{F_1, F_2, \ldots, F_{3n}\}.
\end{aligned}$$

Thus, the edge labels are distinct. Therefore, $K_{1,n} \oslash K_{1,2}$ admits almost super fibonacci graceful labeling. Hence, $K_{1,n} \oslash K_{1,2}$ is an almost super fibonacci graceful graph.

This example shows that the graph $K_{1,3} \oslash K_{1,2}$ is an almost super fibonacci graceful graph.

$K_{1,3} \oslash K_{1,2}$:

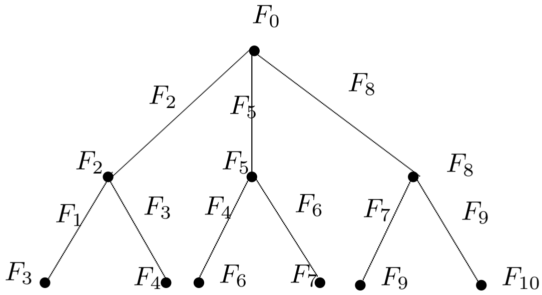

Fig.11

Definition 2.9. Let u, v be the center vertices of $B_{2,n}$. Let u_1, u_2 be the vertices joined with u and $v_1, v_2, \ldots, v_n$ be the vertices joined with v. Let $w_1, w_2, \ldots, w_n$ be the vertices of the subdivision of edges vv_i $(1 \leq i \leq n)$ respectively and it is denoted by $(B_{2,n} : w_i)$, $1 \leq i \leq n$.

Theorem 2.9. The graph $G = (B_{2,n} : w_i)$, $1 \leq i \leq n$, where $n \geq 2$ is an almost super fibonacci graceful graph.

Proof. Let u, v be the center vertices of $B_{2,n}$. Let u_1, u_2 be the vertices joined with u and $v_1, v_2, \ldots, v_n$ be the vertices joined with v and $w_1, w_2, \ldots, w_n$ be the vertices of the subdivision of edges vv_i $(1 \leq i \leq n)$ respectively. Also, $|V(G)| = 2n + 4$ and $|E(G)| = 2n + 3$.

Case(i) : n is odd.

Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u) = F_{2n+2}$, $f(u_1) = F_{2n+4}$, $f(u_2) = F_{2n}$, $f(v) = F_0$, $f(v_{2i-1}) = F_{2n-4i+5}$, $1 \leq i \leq \frac{n+1}{2}$,$f(v_{2i}) = F_{2n-4i}$, $1 \leq i \leq \frac{n-1}{2}$, $f(w_{2i-1}) = F_{2n-4i+3}$, $1 \leq i \leq \frac{n+1}{2}$, $f(w_{2i}) = F_{2n-4i+2}$, $1 \leq i \leq \frac{n-1}{2}$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(uu_i) : 1 \leq i \leq 2\}$.

Then

$$\begin{aligned} E_1 &= \{|f(u) - f(u_i)| : 1 \leq i \leq 2\} \\ &= \{|f(u) - f(u_1)|, |f(u) - f(u_2)|\} \\ &= \{|F_{2n+2} - F_{2n+4}|, |F_{2n+2} - F_{2n}|\} \\ &= \{F_{2n+3}, F_{2n+1}\}. \end{aligned}$$

Let $E_2 = \{f^*(uv)\}$.

Then

$$\begin{aligned} E_2 &= \{|f(u) - f(v)|\} \\ &= \{|F_{2n+2} - F_0|\} \\ &= \{F_{2n+2}\}. \end{aligned}$$

Let $E_3 = \{f^*(vw_{2i-1}) : 1 \leq i \leq \frac{n+1}{2}\}$.

Then

$$\begin{aligned}
E_3 &= \{|f(v)-f(w_{2i-1})| : 1 \le i \le \frac{n+1}{2}\} \\
&= \{|f(v)-f(w_1)|, |f(v)-f(w_3)|, \ldots, |f(v)-f(w_{n-2})|, |f(v)-f(w_n)|\} \\
&= \{|F_0-F_{2n-1}|, |F_0-F_{2n-5}|, \ldots, |F_0-F_5|, |F_0-F_1|\} \\
&= \{F_{2n-1}, F_{2n-5}, \ldots, F_5, F_1\}.
\end{aligned}$$

Let $E_4 = \{f^*(vw_{2i}) : 1 \le i \le \frac{n-1}{2}\}$.
Then

$$\begin{aligned}
E_4 &= \{|f(v)-f(w_{2i})| : 1 \le i \le \frac{n-1}{2}\} \\
&= \{|f(v)-f(w_2)|, |f(v)-f(w_4)|, \ldots, |f(v)-f(w_{n-3})|, |f(v)-f(w_{n-1})|\} \\
&= \{|F_0-F_{2n-2}|, |F_0-F_{2n-6}|, \ldots, |F_0-F_8|, |F_0-F_4|\} \\
&= \{F_{2n-2}, F_{2n-6}, \ldots, F_8, F_4\}.
\end{aligned}$$

Let $E_5 = \{f^*(v_{2i-1}w_{2i-1}) : 1 \le i \le \frac{n+1}{2}\}$.
Then

$$\begin{aligned}
E_5 &= \{|f(v_{2i-1})-f(w_{2i-1})| : 1 \le i \le \frac{n+1}{2}\} \\
&= \{|f(v_1)-f(w_1)|, |f(v_3)-f(w_3)|, \ldots, |f(v_{n-2})-f(w_{n-2})|, |f(v_n)-f(w_n)|\} \\
&= \{|F_{2n+1}-F_{2n-1}|, |F_{2n-3}-F_{2n-5}|, \ldots, |F_7-F_5|, |F_3-F_1|\} \\
&= \{F_{2n}, F_{2n-4}, \ldots, F_6, F_2\}.
\end{aligned}$$

Let $E_6 = \{f^*(v_{2i}w_{2i}) : 1 \le i \le \frac{n-1}{2}\}$.
Then

$$\begin{aligned}
E_6 &= \{|f(v_{2i})-f(w_{2i})| : 1 \le i \le \frac{n-1}{2}\} \\
&= \{|f(v_2)-f(w_2)|, |f(v_4)-f(w_4)|, \ldots, |f(v_{n-3})-f(w_{n-3})|, \\
&\quad |f(v_{n-1})-f(w_{n-1})|\} \\
&= \{|F_{2n-4}-F_{2n-3}|, |F_{2n-8}-F_{2n-6}|, \ldots, |F_6-F_8|, |F_2-F_4|\} \\
&= \{F_{2n-3}, F_{2n-7}, \ldots, F_7, F_3\}.
\end{aligned}$$

Theorefore, $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 = \{F_1, F_2 \ldots F_{2n+3}\}$.

Thus, the edge labels are distinct. Therefore, the graph $(B_{2,n} : w_i)$, $1 \le i \le n$ admits almost super fibonacci graceful labeling. Hence, $(B_{2,n} : w_i)$, $1 \le i \le n$ is an almost super fibonacci graceful graph.

For example the almost super fibonacci graceful labeling of $B_{2,5}$ is shown in Fig. 12.

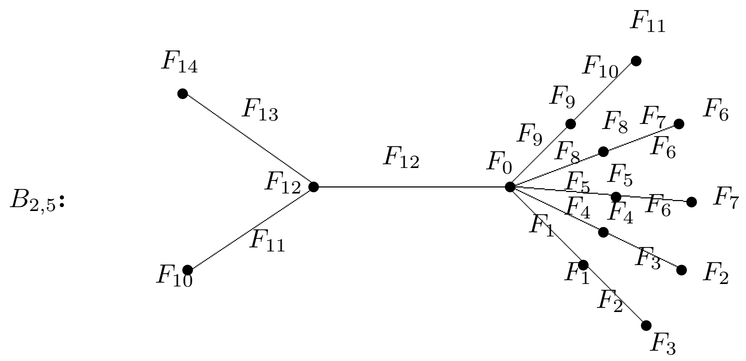

Fig.12

Case(ii) : n is even.

Define $f : V(G) \to \{F_0, F_1, F_2, \ldots, F_{q-1}, F_{q+1}\}$ by $f(u) = F_{2n+2}$, $f(u_1) = F_{2n+4}$, $f(u_2) = F_{2n}$, $f(v) = F_0$, $f(v_n) = F_1$, $f(v_{2i-1}) = F_{2n-4i+5}$, $1 \leq i \leq \frac{n}{2}$, $f(v_{2i}) = F_{2n-4i}$, $1 \leq i \leq \frac{n}{2} - 1$, $f(w_{2i-1}) = F_{2n-4i+3}$, $1 \leq i \leq \frac{n}{2}$, $f(w_{2i}) = F_{2n-4i+2}$, $1 \leq i \leq \frac{n}{2}$.

Next, we claim that the edge labels are distinct.

Let $E_1 = \{f^*(uu_i) : 1 \leq i \leq 2\}$.

Then

$$\begin{aligned} E_1 &= \{|f(u) - f(u_i)| : 1 \leq i \leq 2\} \\ &= \{|f(u) - f(u_1)|, |f(u) - f(u_2)|\} \\ &= \{|F_{2n+2} - F_{2n+4}|, |F_{2n+2} - F_{2n}|\} \\ &= \{F_{2n+3}, F_{2n+1}\}. \end{aligned}$$

Let $E_2 = \{f^*(uv)\}$.

Then

$$\begin{aligned} E_2 &= \{|f(u) - f(v)|\} \\ &= \{|F_{2n+2} - F_0|\} \\ &= \{F_{2n+2}\}. \end{aligned}$$

Let $E_3 = \{f^*(vw_{2i-1}) : 1 \leq i \leq \frac{n}{2}\}$

Then

$$\begin{aligned} E_3 &= \{|f(v) - f(w_{2i-1})| : 1 \leq i \leq \frac{n}{2}\} \\ &= \{|f(v) - f(w_1)|, |f(v) - f(w_3)|, \ldots, |f(v) - f(w_{n-3})|, |f(v) - f(w_{n-1})|\} \\ &= \{|F_0 - F_{2n-1}|, |F_0 - F_{2n-5}|, \ldots, |F_0 - F_7|, |F_0 - F_3|\} \\ &= \{F_{2n-1}, F_{2n-5}, \ldots, F_7, F_3\}. \end{aligned}$$

Let $E_4 = \{f^*(vw_{2i}) : 1 \leq i \leq \frac{n-1}{2}\}$

Then

$$\begin{aligned} E_4 &= \{|f(v) - f(w_{2i})| : 1 \le i \le \frac{n-1}{2}\} \\ &= \{|f(v) - f(w_2)|, |f(v) - f(w_4)|, \ldots, |f(v) - f(w_{n-2})|, |f(v) - f(w_n)|\} \\ &= \{|F_0 - F_{2n-2}|, |F_0 - F_{2n-6}|, \ldots, |F_0 - F_{10}|, |F_0 - F_6|\} \\ &= \{F_{2n-2}, F_{2n-6}, \ldots, F_{10}, F_6\}. \end{aligned}$$

Let $E_5 = \{f^*(v_{2i-1}w_{2i-1}) : 1 \le i \le \frac{n}{2}\}$.
Then

$$\begin{aligned} E_5 &= \{|f(v_{2i-1}) - f(w_{2i-1})| : 1 \le i \le \frac{n}{2}\} \\ &= \{|f(v_1) - f(w_1)|, |f(v_3) - f(w_3)|, \ldots, |f(v_{n-3}) - f(w_{n-3})|, \\ &\quad |f(v_{n-1}) - f(w_{n-1})|\} \\ &= \{|F_{2n+1} - F_{2n-1}|, |F_{2n-3} - F_{2n-5}|, \ldots, |F_9 - F_7|, |F_5 - F_3|\} \\ &= \{F_{2n}, F_{2n-4}, \ldots, F_8, F_4\}. \end{aligned}$$

Let $E_6 = \{f^*(v_{2i}w_{2i}) : 1 \le i \le \frac{n}{2} - 2\}$.
Then

$$\begin{aligned} E_6 &= \{|f(v_{2i}) - f(w_{2i})| : 1 \le i \le \frac{n}{2} - 2\} \\ &= \{|f(v_2) - f(w_2)|, |f(v_4) - f(w_4)|, \ldots, |f(v_{n-4}) - f(w_{n-4})|, \\ &\quad |f(v_{n-2}) - f(w_{n-2})|\} \\ &= \{|F_{2n-4} - F_{2n-3}|, |F_{2n-8} - F_{2n-6}|, \ldots, |F_8 - F_{10}|, |F_4 - F_6|\} \\ &= \{F_{2n-3}, F_{2n-7}, \ldots, F_9, F_5\}. \end{aligned}$$

Let $E_7 = \{f^*(vv_n), f^*(v_nw_n)\}$.
Then

$$\begin{aligned} E_7 &= \{|f(v) - f(v_n)|, |f(v_n) - f(w_n)|\} \\ &= \{|F_0 - F_2|, |F_2 - F_1|\} \\ &= \{F_2, F_1\}. \end{aligned}$$

Theorefore

$$\begin{aligned} E &= E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \\ &= \{F_1, F_2, \cdots, F_{2n+3}\}. \end{aligned}$$

Thus, the edge labels are distinct. Therefore, the graph $(B_{2,n} : w_i)$, $1 \le i \le n$ admits almost super fibonacci graceful labeling. Hence, $(B_{2,n} : w_i)$, $1 \le i \le n$ is an almost super fibonacci graceful graph.

For example the almost super fibonacci graceful labeling of $B_{2,4}$ is shown in Fig. 13.

$B_{2,4}$:

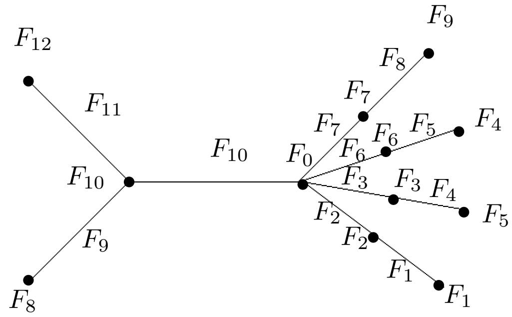

Fig.13

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